LORENTZ AND EINSTEIN
- IN PROPER HISTORICAL PERSPECTIVE*

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Einstein, the author of special relativity, did not define an 'event' explicitly as the crossing of the end points of rods, as we do in our new approach. He states in his book The Meaning of Relativity:

"The experiences of an individual are arranged in a series of events; in this series, the single events which we remember appear to be ordered according to the criterion earlier and later which cannot be analysed further."

Lorentz transformation deals with the separation in space and time of two events. Einstein derives the contraction of length of moving bodies therefrom. On the contrary, we start with Lorentz Contraction as fundamental and derive the Lorentz Transformation by defining events precisely as the crossing of the end points of rods. This eliminates misconceptions and paradoxes like virtual or real contraction, faster than light particles, and differential aging of twins. Actually this is what Lorentz attempted to do but he needlessly invoked the medium of ether which is dragged along with the particle. Though ether is non-existent, the distance between 'equivalent' observers is meaningful and so is its contraction, which is postulated in much the same way as curvature in space is postulated in general relativity.

Axioms of Special Relativity

Special relativity deals with changes in intervals in space and time only in the direction of motion. Therefore we need consider only two coordinates, one-dimensional space x and one dimensional time t based on the following axioms.

AXIOM I: By definition a point treated as an observer is at rest since x and t are measured with respect to it. All observers at rest with respect to it, even if separated in space, are equivalent. If P is an observer, and if $P_1, P_2, P_3, \cdots, P_n$ are at rest with respect to it, then all of them are equivalent observers though separated by distances $L_1, L_2, L_3, \cdots, L_n$:

$$PP_1 = L_1, \quad P_1P_2 = L_2, \quad \cdots, \quad P_{n-1}P_n = L_n.$$

These distances remain the same for all time and are called the lengths of the 'rods' $PP_1, P_1P_2, \cdots, P_{n-1}P_n$.

If a point P moves with velocity v with respect to P, it moves with the same velocity with respect to all P.

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AXIOM II: If we treat Q as the observer, it is by definition at rest and $P, P_1, \cdots, P_n$ are all moving with velocity $-v$ with respect to Q.

If $Q_1, Q_2, \cdots, Q_n$ are points at rest with respect to Q, they are observers equivalent to Q though separated by distances $\ell_1, \ell_2, \cdots, \ell_n$:

$$\ell_1 = QQ_1, \quad \ell_2 = Q_1Q_2, \quad \cdots, \quad \ell_n = Q_{n-1}Q_n.$$

AXIOM III - FUNDAMENTAL TO SPECIAL RELATIVITY: If $P, P_1, \cdots, P_n$ are equivalent observers, the $Q's$ which are 'observed' points as measured by the $P's$ are

$$K\ell_1, \quad K\ell_2, \quad \cdots, \quad K\ell_n,$$

where

$$K = \sqrt{1 - \frac{v^2}{c^2}},$$

is the contraction factor with c as a constant $> v$.

If $Q, Q_1, \cdots, Q_n$ are treated as observers, then the distances between the $P's$ as measures by the $Q's$ are

$$KL_1, \quad KL_2, \quad \cdots, \quad KL_n.$$

These distances measured at the same time remain the same for all time whether the P_i or Q_i are observers or the observed. There is perfect symmetry between the observers and the observed which are in relative motion.

A rod connects two equivalent points and is a continuous set of equivalent observers or observed points.

All these axioms can be summarised by the following aphorism: If P and Q are in relative motion, the spaces (rods) attached to them are in the same relative motion but each is contracted when observed by the other. The universe of 'events' is the intersection of relatively moving spaces (rods).

The key that unlocks space-time unity

With these axioms, Lorentz Transformation and the constancy of the velocity of light follow as natural consequences.

Consider a rod AB of length $x - vt$ moving with velocity v across a stationary rod CD of length x. Pictorially, write AB as a horizontal line with CD another line below it and parrallel to it, with C right below A. Concidences AC and BD are separated by distance x and time t.

If we shift to the rest frame of AB, its length is

$$\frac{x - vt}{\sqrt{1 - (v^2/c^2)}} = x'.$$

The rod CD moving with velocity $-v$ has length

$$x\sqrt{1 - (v^2/c^2)}.$$

The time interval between the coincidences AC and BD is

$$t' = -\frac{1}{v}\{\frac{x - vt}{\sqrt{1 - (v^2/c^2)}} - x\sqrt{1 - (v^2/c^2)}\} = \frac{t - (vx/c^2)}{\sqrt{1 - (v^2/c^2)}},$$

which is the Lorentz Transformation. The constancy of the velocity of light follows as a corollary.

If in Axiom III, we just choose the contraction to be K, and impose the condition that the velocity of light is constant, i.e. if we set $x = ct, x' = ct'$ in

$$t' = -\frac{1}{c}\{\frac{x - vt}{K} - xK\}, \quad \text{and} \quad x' = \frac{x - vt}{K},$$

we get $K = \sqrt{1 - (v^2/c^2)}$.

Corollaries

I) Defining $x/t = V, x'/t' = V'$, we get with $v < c$,

$$V' = \frac{V - v}{1 - (Vv/c^2)}.$$

To interpret the transformation, we consider two cases:

Case 1: $x/t < v$, consequently $x'/t' < c$.

Choosing a rod of length $x - Vt = 0$, we find that it represents a point moving with velocity V across x and V' across x'. Thus it represents the Velocity Transformation formula.

Case 2: If $x/t > c$, we can write it as c^2/v and note that $x - vt > x - ct$. Thus the shortest length of the rod is $x - ct$. Therefore x/t does not represent a velocity but a ratio with the transformation properties of a velocity. The rod of length $x - vt$ moves a distance $(v^2/c^2)x$ in time vx/c^2 to generate events separated by x and t. This eliminates the concept of a Tachyon (faster than light particle) since x/t is NOT a velocity. In the rest system of AB, its length is

$$\frac{x\{1 - (v^2/c^2)\}}{\sqrt{1 - (v^2/c^2)}} = x\sqrt{1 - (v^2/c^2)}.$$

Hence $t' = 0$ (simultaneity).

In general, if $x/t > v$, then x'/t' is also greater as observed by a system travelling with velocity $< c$.

II) If $t = 0$, then $t = -vx'/c^2$ (non-simultaneity).

III) If a rod moving with velocity V has length L, its length in the rest system of a particle (rod) moving with velocity v is

$$\frac{L\sqrt{1 - (V_2^2/c^2)}}{\sqrt{1 - (V_1^2/c^2)}} = \frac{L\{1 + (vV_2)/c^2)\}}{1 - (v^2/c^2)}, \quad V_2 = \frac{V_1 - v}{1 - (vV_1/c^2)}.$$

A rod moving with velocity c and length L has as its altered length

$$\frac{L\sqrt{1+(v/c)}}{\sqrt{1-(v/c)}}.$$

IV) If we put $v = c$ in $x - vt$, the rod of light BA of length $ct - x$ $(x < ct)$ moves across CD generating events separated by x and t.

If $x > ct$, the rod of light $AB - x - ct$ moves to generate the events.

In the rest system of a rod moving with velocity v, the rod of light moves with the same velocity c while CD moves with velocity $-v$. Hence

$$t' = \frac{1}{c+v}\{(ct-x)\sqrt{\frac{1+(v/c)}{1-(v/c)}} + x\sqrt{1-(v^2/c^2)}\} = \frac{t-(vx/c^2)}{\sqrt{1-(v^2/c^2)}}.$$

It is therefore justifiable to call the rod $x - vt$ moving with velocity v as the 'Master Key' to Space-Time unity, since it represents a continuous infinity of rods for varying values of v, any two of which generate events separately by x and t.

Can mathematics be more beautiful?

V) We note that the events x and t can also be generated by a rod of light $ct - x$ moving with velocity c crossing a rod of light $ct + x$ moving with velocity $-c$. Using the changes of length, we obtain the Lorentz Transformation

$$t = \frac{1}{2c}\{\frac{ct-x}{B} + (ct+x)B\} = \frac{t-(vx/c^2)}{K}, \quad B = \sqrt{\frac{c-v}{c+v}}.$$

All these concepts are imbedded in a single mathematical statement that the eigen values of the Lorentz matrix transforming x/c and t

$$\frac{1}{K}\begin{bmatrix} 1 & -v/c \\ -v/c & 1 \end{bmatrix}$$

are B and $1/B$ with eigen vectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

VI) If x and t are the spatial and time intervals representing a point with velocity v crossing a stationary rod $x, (x/t) = v$, to an observer moving with velocity v, the point is stationary but the rod moves with velocity $-v$ and shorter length xK. The events are represented by the shorter rod crossing the stationary point with $x' - 0, t' = tK$, where

$$K = \sqrt{1-(v^2/c^2)}.$$

Historically t is compared with t' and called Time Dilatation while the shorter length is compared with the longer and called Lorentz Contraction. Both represent the same phenomenon. If x and t are the spatial and time intervals representing light crossing a stationary rod x, the velocity of light is $c = (x/t)$. To an observer moving with velocity v, the space and time intervals between the events are x', t' such that $(x'/t') = c$ representing light crossing the rod of length xK moving with velocity $-v$

$$t' = \frac{xK}{c+v} = Bt, \quad x' = Bx.$$

If we replace c by $-c$, then B changes to $1/B$. These results also follow from the Lorentz Transformation with

$$x = ct, \quad x' \frac{\{1-(v/c)\}x}{K} = Bx.$$

We have proved that if a rod of light has length L, then to an observer moving with velocity v, the length is $L' = L/B$. If the rod of light is moving with velocity $-c$, then the length is altered to LB.

The difference between the transformation properties of x and L is not puzzling but is an example of perfect mathematical harmony.

The events separated by x and t can be generated by the crossing of a rod of light $ct - x$ moving with velocity c, and the rod of length $ct + x$ moving with velocity $-c$. If we now assume that the events represent the motion of light (rod length $= 0$) moving against a rod of light of length $2x$ moving with velocity $-c$, then to an observer moving with velocity v, the length is contracted by a factor B characterising the transformation property of x and t. Note that $2x$ is a length while x is the spatial separation between the events with time separation.

VII. The change in the length of the rod of light can be obtained by another method which amounts to a derivation of the velocity transformation.

Consider the rod of light of length L as observed by a point O. If O' is moving with velocity v, then O overtakes O' in time $L/(c-v)$, where $c - v$ is the external relative velocity between light and O' which moves a distance $vL/(c-v)$ across a stationary rod of the same length. In the rest system of O', this rod is contracted by a factor K and moves with velocity$-v$ crossing the stationary point O' in time $KL/(c-v)$. This is also the time for the rod of light to cross the stationary point O' and so the rod of light has length $L' = cKL/(c-v) = L/B$.

The same argument can be applied to a rod of length L moving with velocity $V < c$ as observed by O, and V' as observed by O'. This leads to

$$L' = \frac{VKL}{V-v}, \quad L = \frac{VKL'}{V'+v}, \quad K^2 = \frac{(V-v)(V'+v)}{VV'},$$

which is the velocity transformation formula in a 'different garb' obtained directly from Lorentz Contraction without going through the transformation. Hence we can write

$$K^2 = (1 - \frac{v}{V})(1 + \frac{v}{V'}) = (1 - \frac{vV}{c^2})(1 + \frac{vV'}{c^2}).$$

This equality which is true for $v, V, V' < c$ also implies that we can replace V by c^2/V and V'' by c^2/V' which are greater than c! This tantalising asymmetry stares us in the face! In fact we can replace any two of the quantities v, V, V' by $c^2/v, c^2/V, c^2/V'$ - an asymmetry with a symmetry!! Finally, defining $V'' - -v$, we can write the velocity transformation as

$$\frac{v}{c} + \frac{V}{c} + \frac{V'}{c} + \frac{cVV'}{c^3} = 0.$$

permitting replacement of any two of $v/c, V/c, V'/c$ by their reciprocals.

Spectacular Symmetries

God said "Let there be Light and there was Light".No one except the creator can explain why Light has the incredible proprty of the constancy of velocity independent of the motion of the observer. The genius of Einstein and Lorentz could explain only the consequence of this property as Space-Time unity.

This unity is expressed in the Lorentz transformation by the inherent spectacular symmetry between the 'transit time' $(x/c) = T$ and the 'basic time' t. This is demonstrated by writing the fundamental equation for a rod $x - vt$ crossing x as

$$\frac{-vt'}{c} = T' - TK, \quad \frac{+vt}{c} = T - T'K.$$

We can write the adjoint equations as

$$\frac{vT'}{c} = t' - tK, \quad \frac{+vT}{c} = t - t'K.$$

The adjoint equation has an equally striking interpretation if we recognise that $t' - tK$ is the non-simultaneity Δ of events which are simultaneous when the rod $x - vt$ with velocity v moves across the stationary part $x - vt$ of the stationary rod x. Hence $cT' = x' = x - (\Delta c^2)/v$.

These are the exotic forms (unnoticed since 1905) of the Lorentz Transformation and its inverse:

$$t' = \{t - (v/cT)\}/K, \quad t = \{t' + (v/cT')\}/K,$$

and

$$T' = \{T - (v/ct)\}/K, \quad T = \{T' + (v/cT'\}/K.$$

In each set, only two equations are independent which implies that if we assume any two of the set T, t, T', t' as known, the other two are determined by the equations.

The perfect symmetry between transit and basic times T and t suggests a method of combining space-time and time-like intervals corresponding to $T^2 - t^2 = \pm a^2$ into one scheme by just requiring $(T^2 - t^2)^2 = a^4$. Defining $P = T^2 + t^2, Q = 2Tt, P^2 - Q^2 = a^4$, we can rewrite

$$\begin{bmatrix} P' \\ Q' \end{bmatrix} = \frac{1}{\sqrt{1 - (V^2/c^2)}} \begin{bmatrix} 1 & -V/c \\ -V/c & 1 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \frac{1}{K^2} \begin{bmatrix} 1 & -v/c \\ -v/c & 1 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}$$

with

$$V = \frac{2v}{1 + (v^2/c^2)}, \quad K^2 = 1 - (v^2/c^2).$$

If we assume that the events are separated by x and t such that $(x/t) = v < c$ representing a point moving with velocity v crossing a stationary rod x, then

$$\frac{T}{t} = \frac{v}{c}, \quad P' = t^2\{1 - (v^2/c^2)\} = a^2 = P\sqrt{1 - (V^2/c^2)}, \quad Q' = 0, \quad t' = tk, \quad T' = 0.$$

If we assume a rod of length $x\{1 - (v^2/c^2)\}$ moving with velocity v across a stationary rod $x, x/t$ with $(c^2/v) > c$, then

$$\frac{t}{T} = \frac{v}{c}, \quad P' = T^2\{1 - (v^2/c^2)\} = a^2 = P\sqrt{1 - (v^2/c^2)}, \quad Q' = 0, \quad T' = Tk, \quad t' = 0,$$

including concepts of Time Dilation and Lorentz Contraction in the same scheme. Here $P^2 - Q^2$ is positive and so $P/Q = c/v$.

The source of all these symmetries is the fundamental circulant matrix

$$\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$$

with eigen values $1 + a$ and $1 - a$ and determinant $(1 - a^2)$. The n^{th} power of this matrix is also a circulant

$$\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}^n = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

with determinant $A^2 - B^2 = (1 - a^2)^n$, with A and B given by

$$A = \frac{1}{2}\{(1 + a)^n + (1 - a)^n\}, \quad B = \frac{1}{2}\{(1 + a)^n - (1 - a)^n\}.$$

We recognise that in special relativity, a is the relative velocity (normalised to the velocity of light) in successive frames of reference, and B/A the compounded relative velocity. If the velocities are different - $a_1, a_2, \cdots, a_n$, we have to merely replace the powers by the products

$$(1 + a_1)(1 + a_2)\cdots(1 + a_n), \quad (1 - a_1)(1 - a_2)\cdots(1 - a_n),$$

in defining A and B. The Lorentz matrices are obtained by dividing each matrix by the root of the corresponding determinant.

One gaps with wonder what immortal hands can frame such perfect symmetry.