A MULTI-DIMENSIONAL EXTENSION OF SYLVESTER'S IDENTITY

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Abstract: *We obtain by combinatorial means, a multi-dimensional extension of Sylvester's famous identity of 1882 that generalized Euler's Pentagonal Numbers Theorem. We also provide a purely q-hypergeometric proof.*

1. Introduction

Although partitions are combinatorial objects, Euler who founded the theory of partitions in the mid-eighteenth century, did not use combinatorial methods but preferred formal power series and generating function techniques. It was Sylvester in the late nineteenth century who first exploited combinatorial methods to study partitions and extended many results of Euler. An important result of Sylvester in his classic paper [10] of 1882 is:

$$(-aq)_\infty = 1 + \sum_{k=1}^{\infty} \frac{a^k q^{(3k^2-k)/2}(-aq)_{k-1}(1+aq^{2k})}{(q)_k}. \tag{1.1}$$

Here and in what follows we have used the standard notation

$$(a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad \text{and} \quad (a)_\infty = \lim_{n\to\infty}(a)_n \quad \text{when} \quad |q|<1. \tag{1.2}$$

The case $a=-1$ in (1.1) yields Euler's famous Pentagonal Numbers Theorem:

$$(q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2-k)/2}. \tag{1.3}$$

The product on the left in (1.1) is the generating function of partitions into distinct parts. Sylvester proved (1.1) combinatorially by analyzing the Ferrers graphs of partitions into distinct parts in terms of their Durfee squares. He then posed a challenge to find a purely q-series proof. Such a proof was found by Cayley. This story is nicely described by Andrews [7].

Our goal here is to analyze partitions into parts occurring in r colors a_1, a_2, .., a_r, where parts of the same color do not repeat, and to prove identity (4.8) below which is an expansion for the product

$$(-a_1q)_\infty(-a_2q)_\infty ... (-a_rq)_\infty, \tag{1.4}$$

which is the generating function for the colored partitions. We prove the multi-dimensional extension (4.8) of (1.1) combinatorially in section 5 by following and extending Sylvester's method and studying the Ferrers graphs of these colored partitions via their Durfee squares. Although Sylvester's identity is classical, this multi-dimensional extension is new.

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In the two dimensional case, our extension of (1.1) is

$$\sum_{k\geq 0} q^{k^2}(-aq)_k(-bq)_k \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j} . \frac{(1+aq^{k+i}+bq^{k+i+j}+abq^{2k+i+j})}{(1+aq^k)(1+bq^k)}$$
$$= (-aq)_\infty(-bq)_\infty, \tag{1.5}$$

Even though $i+j=k$, we have deliberately written $k+i+j$ and $2k+i+j$ in (1.5) for the purpose of the r-dimensional extension below. We first provide the combinatorial proof of (1.5) in section 2 which will help in a clearer understanding of the combinatorial proof of the r-dimensional identity (4.8) presented in Section 5. Upon seeing (1.5), Andrews [8] provided a purely q-series proof of it (see section 3) by extending Cayley's proof of (1.1). We are able to extend Andrews' method to establish (4.8) by purely q-series means (see Section 6).

2. The two dimensional Sylvester identity: combinatorial derivation

We view

$$(-aq)_\infty(-bq)_\infty \tag{2.1}$$

as the generating functions for partitions occurring in two colors a and b, with parts of the same color not repeating. We use a, b to denote the colors as well as the parameters that keep track of the number of parts in the colors. We assume $a < b$ as the order among the colors so as to discuss partiitions into parts in the two colors. When we draw the Ferrers graphs of these two colored partitions, we color only the last node on the right of each row as either a or b; the remaining nodes are uncolored.

The *Durfee square* D in a Ferrers graph of a partition π is the largest square of nodes starting from the top left hand corner of the graph. The portion to the right of the Durfee square represents a partition which we denote by π_r. Similarly, the portion below the Durfee square is a partition denoted by π_b.

When considering Ferrers graphs of two colored partitions generated by the product (2.1) having a $k \times k$ Durfee square D, four cases arise:

Case 1: The bottom right node in D is uncolored

Thus all nodes in D are uncolored. So the contribution of the D to the generating function in this case is simply

$$q^{k^2}. \tag{2.2}$$

The portion π_b could have parts up to size k. So its generating function is

$$(-aq)_k(-bq)_k. \tag{2.3}$$

The portion π_r to the right of D has exactly k parts of which i could be in color a and j in color b with $i+j=k$. So the generating function of the partitions π_r is

$$\sum_{i=0}^{k} \frac{a^i b^{k-i} q^{T_i+T_{k-i}}}{(q)_i(q)_{k-i}} = \sum_{i+j=k} \frac{a^i b^j q^{T_i+T_j}}{(q)_i(q)_j}, \tag{2.4}$$

where $T_m = m(m+1)/2$ is the m-th triangular number. Thus the generating function for Case 1 is

$$q^{k^2}(-aq)_k(-bq)_k \sum_{i+j=k} \frac{a^i b^j q^{T_i+T_j}}{(q)_i(q)_j}. \tag{2.5}$$

Case 2: The bottom right node in D has color b, rest in D uncolored
In this case the contribution of D to the generating function is

$$bq^{k^2}. \tag{2.6}$$

Since the bottom right node of D has color b, if the largest part of π_b has size k, then it must have color a. There is no such restriction on the colors for the parts of π_b which are less than k. Thus the generating function of π_b in this case is

$$(-aq)_k(-bq)_{k-1}. \tag{2.7}$$

We note that π_r will have exactly $k-1$ parts. So its generating function is

$$\sum_{i=0}^{k-1} \frac{a^i b^{k-i-1} q^{T_i+T_{k-i-1}}}{(q)_i(q)_{k-i-1}} \tag{2.8}$$

Thus the generating function for Case 2 is

$$bq^{k^2}(-aq)_k(-bq)_{k-1} \sum_{i=0}^{k-1} \frac{a^i b^{k-i-1} q^{T_i+T_{k-i-1}}}{(q)_i(q)_{k-i-1}},$$

which can be written in the form

$$q^{k^2}(-aq)_k(-bq)_{k-1} \sum_{i+j=k, j\geq 1} \frac{a^i b^j q^{T_i+T_{j-1}}}{(q)_i(q)_{j-1}}. \tag{2.9}$$

similar to (2.5).

Case 3: The bottom right node in D has color a, rest in D uncolored
This is similar to Case 2 with the following difference: Since $a < b$ is the order of the colors, the partition π_b will not have a part of size k. This all parts of π_b will be $\leq k-1$. So the generating function of Case 3 will be

$$q^{k^2}(-aq)_{k-1}(-bq)_{k-1} \sum_{i+j=k, i\geq 1} \frac{a^i b^j q^{T_{i-1}+T_j}}{(q)_{i-1}(q)_j}. \tag{2.10}$$

Finally we have

Case 4: The bottom right node in D has color a, the node above it has color b, and the rest in D uncolored
In this case the contribution of D to the generating function is

$$abq^{k^2}.$$

The generating function of π_b is the same as in Case 3. The main difference is that the partition π_r will now have exactly $k-2$ parts. So the generating function of Case 4 is

$$abq^{k^2}(-aq)_{k-1}(-bq)_{k-1}\sum_{i=0}^{k-2}\frac{a^ib^{k-i-2}q^{T_i+T_{k-i-2}}}{(q)_i(q)_{k-i-2}},$$

which can be written in the form

$$q^{k^2}(-aq)_{k-1}(-bq)_{k-1}\sum_{i+j=k,i\geq 1,j\geq 1}\frac{a^ib^jq^{T_{i-1}+T_{j-1}}}{(q)_{i-1}(q)_{j-1}}. \tag{2.11}$$

similar to (2.5).

We need to add the generating functions in (2.5), (2.9), (2.10), (2.11), sum this over k and add 1 (for the null partition) to get

$$\begin{aligned}1+\sum_{k\geq 1}q^{k^2}(-aq)_k(-bq)_k\sum_{i+j=k}\frac{a^ib^jq^{T_i+T_j}}{(q)_i(q)_j}\\ +\sum_{k\geq 1}q^{k^2}(-aq)_k(-bq)_{k-1}\sum_{i+j=k,j\geq 1}\frac{a^ib^jq^{T_i+T_{j-1}}}{(q)_i(q)_{j-1}}\\ +\sum_{k\geq 1}q^{k^2}(-aq)_{k-1}(-bq)_{k-1}\sum_{i+j=k,i\geq 1}\frac{a^ib^jq^{T_{i-1}+T_j}}{(q)_{i-1}(q)_j}\\ +\sum_{k\geq 1}q^{k^2}(-aq)_{k-1}(-bq)_{k-1}\sum_{i+j=k,i\geq 1,j\geq 1}\frac{a^ib^jq^{T_{i-1}+T_{j-1}}}{(q)_{i-1}(q)_{j-1}}=1+I+II+III+IV\end{aligned} \tag{2.12}$$

We group the terms as follows. For fixed $k\geq 1$, consider first the sum of the terms in I and III:

$$\begin{aligned}&q^{k^2}(-aq)_k(-bq)_k\sum_{i+j=k}\frac{a^ib^jq^{T_i+T_j}}{(q)_i(q)_j}+q^{k^2}(-aq)_{k-1}(-bq)_{k-1}\sum_{i+j=k,i\geq 1}\frac{a^ib^jq^{T_{i-1}+T_j}}{(q)_{i-1}(q)_j}\\ &=q^{k^2}(-aq)_{k-1}(-bq)_{k-1}\{(1+aq^k)(1+bq^k)\sum_{i+j=k}\frac{a^ib^jq^{T_i+T_j}}{(q)_i(q)_j}+\sum_{i+j=k,i\geq 1}\frac{a^ib^jq^{T_{i-1}+T_j}}{(q)_{i-1}(q)_j}\}.\end{aligned} \tag{2.13}$$

Note that in (2.13) the condition $i\geq 1$ is redundant because by definition $(q)_{-1}^{-1}=0$. Thus the expression in (2.13) is

$$=q^{k^2}(-aq)_{k-1}(-bq)_{k-1}\sum_{i+j=k}\frac{a^ib^jq^{T_{i-1}+T_j}}{(q)_i(q)_j}\{(1+aq^k)(1+bq^k)q^i+(1-q^i)\}$$

$$=q^{k^2}(-aq)_{k-1}(-bq)_{k-1}\sum_{i+j=k}\frac{a^ib^jq^{T_{i-1}+T_j}}{(q)_i(q)_j}(1+aq^{k+i}+bq^{k+i}+abq^{2k+i}). \tag{2.14}$$

Similarly, for fixed $k \geq 1$, the sum of the terms in II and IV is

$$q^{k^2}(-aq)_{k-1}(-bq)_{k-1} \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i (q)_{j-1}} \{(1+aq^k)q^i + (1-q^i)\}$$

$$= q^{k^2}(-aq)_{k-1}(-bq)_{k-1} \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i (q)_{j-1}} (1+aq^{k+i}). \tag{2.15}$$

Finally, summing the expressions in (2.14) and (2.15) we get

$$q^{k^2}(-aq)_{k-1}(-bq)_{k-1} \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i (q)_j}$$

$$\times\{(1+aq^{k+i}+bq^{k+j}abq^{2k+i})q^j + (1+aq^{k+i})(1-q^j)\}$$

$$= q^{k^2}(-aq)_{k-1}(-bq)_{k-1} \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i (q)_j} (1+aq^{k+i}+bq^{k+i+j}+abq^{2k+i+j}). \tag{2.16}$$

Thus summing the expression in (2.16) over $k \geq 1$ and adding 1 we get

$$1+\sum_{k\geq 1} q^{k^2}(-aq)_{k-1}(-bq)_{k-1} \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i (q)_j} (1+aq^{k+i}+bq^{k+i+j}+abq^{2k+i+j})$$

$$= (-aq)_\infty(-bq)_\infty. \tag{2.17}$$

This is equivalent to (1.5) which is written as a sum over $k \geq 0$ by absorbing the starting term 1 which corresponds to $k = 0$ into the summation. This proves the two dimensional extension (1.5) of Sylvester's identity.

Remark: Although the product in (1.5) and (2.17) is symmetric in a and b, this symmetry is not explicitly seen in the series. There is the trivial series expansion

$$(-aq)_\infty(-bq)_\infty = (\sum_i \frac{a^i q^{T_i}}{(q)_i})(\sum_j \frac{b^j q^{T_j}}{(q)_j}) = \sum_{i,j} \frac{a^i b^j q^{T_i+T_j}}{(q)_i (q)_j}$$

obtained by straightforward multiplication. But a more interesting expansion on the product in (1.5) symmetric in a and b was obtained by Alladi-Gordon [3] which was a generalization and refinement of Schur's partition theorem [9]. More specifically, the symmetric *key identity* for the generalized Schur theorem derived in [3] is:

$$(-aq)_\infty(-bq)_\infty = \sum_{i,j} \frac{a^i b^j q^{T_i+T_j}}{(q)_i (q)_j} = \sum_{i,j} a^i b^j \sum_{i=r+t, j=s+t} \frac{q^{T_{r+s+t}+T_t}}{(q)_r (q)_s (q)_t}. \tag{2.18}$$

Andrews observed that it is possible to write down a symmetric version of (1.5). This is given in the next section after describing Andrews' q-theoretic proof of (1.5).

3. The two dimensional Sylvester identity: q-series proof

Our derivation of the two dimensional identity combinatorially in the previous section will motivate the combinatorial derivation of the r-dimensional version in Section 5. But before that it will be instructive to provide here Andrews' q-theoretic proof of (1.5) [5] which will help us extend the underlying ideas to provide a q-theoretic proof of the r-dimensional identity ().

Andrews' Proof: Begin by writing the series in (1.5) as

$$\sum_{k\geq 0} q^{k^2}(-aq)_k(-bq)_k \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j} . \frac{Y}{(1+aq^k)(1+bq^k)}, \tag{3.1}$$

where

$$Y = Y(a,b) = 1 + aq^{k+i} + bq^{k+i+j} + abq^{2k+i+j}, \quad \text{with} \quad k = i+j. \tag{3.2}$$

Next multiply both sides of (1.5) by $(1+a)(1+b)$ to rewrite it as

$$\sum_{k\geq 0} q^{k^2}(-a)_k(-b)_k \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j} .Y = (-a)_\infty(-b)_\infty. \tag{3.3}$$

Now denote the product on the right in (3.3) by $P(a,b)$, and the sum on the left in (3.3) by $S(a,b)$. The product clearly satisfies the functional equation

$$P(a,b) = (1+a)(1+b)P(aq,bq).$$

Following Cayley's method it will be shown below that

$$S(a,b) = (1+a)(1+b)S(aq,bq). \tag{3.4}$$

From this (1.5) will follow because P and S satisfy the same initial conditions.

Now note that Y can be rewritten as

$$\begin{aligned} Y &= q^k(1+aq^k)(1+bq^k) + aq^{k+i}(1-q^j) + (1-q^k) \\ &= q^k(1+aq^k)(1+bq^k) + q^i(1+aq^k)(1-q^j) + (1-q^i) =: Z. \end{aligned} \tag{3.5}$$

Using (3.5), we may rewrite $S(a,b)$ as

$$\begin{aligned} S(a,b) = &\sum_{k\geq 0} q^{k^2}(-a)_{k+1}(-b)_{k+1} \sum_{i+j=k} \frac{(aq)^i (bq)^j q^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j} \\ &+ \sum_{k\geq 1} q^{k^2}(-a)_{k+1}(-b)_k \sum_{i+j=k} \frac{(aq)^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i(q)_{j-1}} \\ &+ \sum_{k\geq 1} q^{k^2}(-a)_k(-b)_k \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_{i-1}(q)_j}. \end{aligned} \tag{3.6}$$

At this stage replace $k \mapsto k+1$ and $j \mapsto j+1$ in the second double sum on the right in (3.6), and similarly replace $k \mapsto k+1$ and $i \mapsto i+1$ on the third double sum in (3.6). Then (3.6) can be rewritten as

$$S(a,b)=\sum_{k\geq 0} q^{k^2}(-a)_{k+1}(-b)_{k+1}\sum_{i+j=k}\frac{(aq)^i(bq)^j q^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j}$$

$$+\sum_{k\geq 0} q^{(k+1)^2}(-a)_{k+2}(-b)_{k+1}\sum_{i+j=k}\frac{(aq)^i(bq)^j.bq^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j}$$

$$\sum_{k\geq 0} q^{(k+1)^2}(-a)_{k+1}(-b)_{k+1}\sum_{i+j=k}\frac{(aq)^i.ab^j q^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j}. \tag{3.7}$$

Finally, we may write the entire expression on the right in (3.7) as

$$=(1+a)(1+b)\sum_{k\geq 0} q^{k^2}(-aq)_k(-bq)_k\sum_{i+j=k}\frac{(aq)^i(bq)^j q^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j}.W, \tag{3.8}$$

where

$$\begin{aligned} W &= 1+q^{2k+1}(1+aq^{k+1})b+q^{2k+1}aq^{-j}\\ &= 1+bq^{2k+1}+abq^{3k+2}+aq^{k+i+1}=Y(aq,bq), \end{aligned} \tag{3.9}$$

with Y given by (3.2) and because $i+j=k$. Thus if (3.9) is combined with (3.8), we see that the functional equation (3.4) is satisfied by S. Since P and S satisfy the same initial conditions, this proves (1.5).

Symmetric version of (1.5): Andrews observed that instead of the bq^{k+i+j} term in (1.5), if we had bq^{k+j}, then it would be symmetric in a and b, and we would have the symmetric product $(1+aq^{k+i})(1+bq^{k+j})$ in the numerator. With this in mind, rewrite (1.5) as

$$\sum_{k\geq 0} q^{k^2}(-aq)_k(-bq)_k\sum_{i+j=k}\frac{a^ib^jq^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j}\frac{(1+aq^{k+i})(1+bq^{k+j})}{(1+aq^k)(1+bq^k)}\quad +$$

$$\sum_{k\geq 0} q^{k^2}(-aq)_k(-bq)_k\sum_{i+j=k}\frac{a^ib^jq^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j}\frac{(bq^{k+i+j}-bq^{k+j})}{(1+aq^k)(1+bq^k)}=(-aq)_\infty(-bq)_\infty. \tag{3.10}$$

Next note that

$$bq^{k+i+j}-bq^{k+j}=-bq^{k+j}(1-q^i). \tag{3.11}$$

So with (3.11), we may rewrite (3.10) as

$$\sum_{k\geq 0} q^{k^2}(-aq)_k(-bq)_k\sum_{i+j=k}\frac{a^ib^jq^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j}\frac{(1+aq^{k+i})(1+bq^{k+j})}{(1+aq^k)(1+bq^k)}\quad -$$

$$\sum_{k\geq 0} q^{k^2}(-aq)_k(-bq)_k\sum_{i+j=k}\frac{a^ib^{j+1}q^{T_{i-1}+T_{j-1}+k+j}}{(q)_{i-1}(q)_j(1+aq^k)(1+bq^k)}=(-aq)_\infty(-bq)_\infty. \tag{3.12}$$

Note that in the second sum on the right in (3.12), only $i \geq 1$ will make a contribution because $1/(q)_{-1} = 0$. So we replace i by $i+1$ in the second sum on the right in (3.12) to rewrite it as

$$\sum_{k\geq 0} q^{k^2}(-aq)_k(-bq)_k \sum_{i+j=k} \frac{a^i b^j q^{T_{i-1}+T_{j-1}}}{(q)_i(q)_j} . \frac{(1+aq^{k+i})(1+bq^{k+j})}{(1+aq^k)(1+bq^k)} \quad -$$

$$\sum_{k\geq 1} q^{k^2}(-aq)_k(-bq)_k \sum_{i+j+1=k} \frac{a^{i+1}b^{j+1}q^{T_i+T_j+k}}{(q)_i(q)_j(1+aq^k)(1+bq^k)} = (-aq)_\infty(-bq)_\infty. \tag{3.13}$$

which is a symmetric version of (1.5).

It would be worthwhile to see whether (1.5) (or (3.13)) is related or can be transformed to (2.18). This is related to Problem 2 stated in Section 7.

4. The multi-dimensional identity

We actually constructed the three dimensional identity (4.1) below that extended (1.5) and saw a pattern in its structure that helped us write down the multi-dimensional identity (4.8). To avoid repetition in the exposition, we will only state the three dimensional identity here but will not provide the details of the q-theoretic proof or the combinatorial derivation of it. Instead we will straightaway provide the combinatorial derivation of the r-dimensional identity (4.8) in Section 5, and its q-theoretic proof in Section 6.

Our three dimensional extension of Sylvester's identity is

$$\sum_{\ell.ge0} q^{\ell^2}(-aq)_\ell(-bq)_\ell(-cq)_\ell \sum_{i+j+k=\ell} \frac{a^i b^j c^k q^{T_{i-1}+T_{j-1}+T_{k-1}}.Y}{(q)_i(q)_j(q)_k(1+aq^\ell)(1+bq^\ell)(1+cq^\ell)}$$

$$= (-aq)_\infty(-bq)_\infty(-cq)_\infty, \tag{4.1}$$

where

$$Y = 1 + aq^{\ell+i} + bq^{\ell+i+j} + cq^{\ell+i+j+k} + abq^{2\ell+i+j} + acq^{2\ell+i+j+k}$$

$$+bcq^{2\ell+i+j+k} + abcq^{3\ell+i+j+k}. \tag{4.2}$$

As in (1.5), here too in (4.2), even though $\ell = i+j+k$, we have preferred to write the exponents of q in a certain form in order to see a pattern. The construction of (4.1) involved computing generating functions in eight different cases, just as the combinatorial construction of (1.5) given in Section 2 involved four cases; the addition of these eight generating functions resulted in an amalgamation and in the expression Y given in (4.2).

In the r-dimensional case, we have parts occurring in r possible colors a_1, a_2, ..., a_r. We need an ordering among the colors, that is an ordering among all colored versions of the same integer n, and for this purpose we take

$$a_1 < a_2 < a_3 < ... < a_r. \tag{4.3}$$

What (4.3) means is that if an integer n occurs in color a_i and in color a_j, then n in color a_j is larger if $j > i$. If $m < n$ are distinct (uncolored) integers, then that order is preserved no matter what colors are assigned to m and n. The symbols a_i will play a dual role: on the one hand they represent colors, on the other they will be free parameters whose powers will represent the number of parts in that color. With these conventions, we are considering partitions into parts in which no integer in a specified color can occur more than once. The generating function for such partitions is obviously

$$(-a_1q)_\infty(-a_2q)_\infty...(-a_rq)_\infty. \tag{4.4}$$

The problem now is to produce an expansion for this product that extends (1.5) and (4.1). To achieve this, we will define the colors a_i to be *primary* colors, the products a_ia_j with $i < j$ to be *secondary* colors, the products $a_ia_ja_k$ with $i < j < k$ to be *ternary* colors, and so on. Given a color x (primary, or secondary, or ternary, or...), we define

$$\ell(x) = \text{the highest order color in} \quad x, \tag{4.5}$$

and

$$\nu(x) = \text{the number of primary colors in} \quad x. \tag{4.6}$$

For example, if $x = a_1a_3a_4$, then $\ell(x) = a_4$, and $\nu(x) = 3$. We now define the polynomial

$$Y = Y_{i_1,i_2,...,i_r}(a_1, a_2, ..., a_r; N) =: \sum_x xq^{N\nu(x)+\sum_{a_j\le\ell(x)} i_j}, \tag{4.7}$$

where the sum is over all possible colors x formed from a_1, a_2, ..., a_r, including the null color, in which case the exponent of q will be zero. With these definitions, the r-dimensional extension of Sylvester's identity is:

$$1 + \sum_{N\ge1} q^{N^2} \prod_{j=1}^{r}(-a_jq)_{N-1} \sum_{i_1+i_2+...i_r=N} \frac{{a_1}^{i_1}{a_2}^{i_2}...{a_r}^{i_r}q^{T_{i_1-1}+T_{i_2-1}+...+T_{i_r-1}}.Y}{(q)_{i_1}(q)_{i_2}...(q)_{i_r}}$$
$$= (-a_1q)_\infty(-a_2q)_\infty...(-a_rq)_\infty, \tag{4.8}$$

where Y is as in (4.7).

Remark: For convenience in the proofs of (4.8) to be given in the next two sections, we prefer to use $(-a_iq)_{N-1}$ instead of $(-a_iq)_N/(1 + a_iq^N)$ as in (1.5) and (4.1). Also to facilitate the proof of (4.8), it is useful to replace Y by

$$Z =: \sum_{j=1}^{r+1}\{\prod_{k=1}^{j-1}(1 + a_kq^N).(1 - q^{i_j}).q^{\sum_{k=1}^{j-1} i_k}\}. \tag{4.9}$$

In defining Z by (4.9), we have formally set $i_{r+1} = \infty$ so that $1-q^{i_{r+1}} = 1$. The replacement of Y by Z in the 2-dimensional case, is the replacement of Y by the expression on the right in (3.5). We need to show that Y and Z are equal and we conclude this section by proving this:

Lemma:

$$Y = Z.$$

Proof of the Lemma: Rewrite the expression in (4.9) as

$$Z = \sum_{j=1}^{r}\{\prod_{k=1}^{j-1}(1+a_kq^N).[q^{\sum_{k=1}^{j-1} i_k} - q^{\sum_{k=1}^{j} i_k}]\} + \prod_{k=1}^{r}(1+a_kq^N)q^{\sum_{k=1}^{r} i_k}$$

$$= 1 + \sum_{j=1}^{r}\{\prod_{k=1}^{j}(1+a_kq^N) - \prod_{k=1}^{j-1}(1+a_kq^N)\}q^{\sum_{k=1}^{j} i_k}$$

$$= 1 + \sum_{j=1}^{r} a_jq^N \prod_{k=1}^{j-1}(1+a_kq^N).q^{\sum_{k=1}^{j} i_k} \tag{4.10}$$

$$= 1 + \sum_{j=1}^{r} \sum_{\ell(x)=a_j} xq^{N\nu(x)+\sum_{a_k \le \ell(x)} i_k} = \sum_{x} xq^{N\nu(x)+\sum_{a_k \le \ell(x)} i_k} = Y,$$

thereby proving the lemma.

5. Combinatorial proof of the multi-dimensional identity

Starting with the product side of (4.8), we will now construct the series in (4.8) combinatorially, and thus provide a combinatorial proof of the multi-dimensional identity.

The infinite product on the right in (4.8) is obviously the generating function of partitions whose parts occur in the colors a_1, ..., a_r, with no part in the same color repeating. We may represent these partitions as Ferrers graphs, where we can use (4.3) to decide the order among parts equal in size but of different colors, and we can color the right extreme node of each row of the graph in the color of the part. We now consider the Durfee squares in such Ferrers graphs and note that there are 2^r cases to consider, namely when the Durfee square D has all uncolored nodes in the right hand extreme column, or the right hand extreme column could consist of nodes colored a_{i_1}, a_{i_2}, ..., a_{i_s}, from the bottom upwards consecutively, with any remaining nodes being uncolored. These cover $2^r - 1$ cases and so there are a total of 2^r cases to consider. We will now show that these 2^r cases can be collected together to form $r+1$ groupings. In other words, the generating functions for these cases can be combined to amalgamate nicely such that in the end there are $r+1$ terms to sum, and this will correspond to the $r+1$ terms in the expression Z we considered previously in (4.9).

Consider all Ferrers graphs which have an $N \times N$ Durfee square and have i_1 parts in color a_1, i_2 parts in color a_2, ..., i_r parts in color a_r, in the portion on and to the right of the Durfee square. This implies that $i_1 + i_2 + ... + i_r = N$. In computing the generating functions for the 2^r cases, the expression

$$C =: q^{N^2} \prod_{j=1}^{r} (-a_j q)_{N-1} \frac{{a_1}^{i_1} {a_2}^{i_2} ... {a_r}^{i_r} q^{T_{i_1-1}+T_{i_2-1}+...+T_{i_r-1}}}{(q)_{i_1-1}(q)_{i_2-1}...(q)_{i_r-1}} \tag{5.1}$$

will occur as a common factor. Thus in discussing the generating functions of the various cases, we focus only on factors other than those contained in C. We call these as the *extra factors* and denote them by E. Of course E will depend on the case being discussed.

Consider first the case c_0 when all nodes in the right hand extreme column of D are uncolored. In this case

$$c_0: \quad E = \frac{(1+a_1q^N)(1+a_2q^N)...(1+a_rq^N)q^{i_1+i_2+...+i_r}}{(1-1q^{i_1})(1-q^{i_2})...(1-q^{i_r})}. \tag{5.2}$$

Next consider the case c_r where the right hand bottom corner of D has color a_r. Since a_r is the highest order color, this implies that all nodes above a_r in D must be uncolored. Thus the extra factors in this case are given by

$$c_r: \quad E = \frac{(1+a_1q^N)(1+a_2q^N)...(1+a_{r-1}q^N)q^{i_1+i_2+...i_{r-1}}(1-q^{i_r})}{(1-q^{i_1})(1-q^{i_2})...(1-q^{i_r})}. \tag{5.3}$$

Note that in (5.3) we have deliberately not canceled the factor $(1-q^{i_r})$ so as to provide the same common denominator for all cases.

From now on the groupings come into play.

Consider now the situation where the bottom right hand corner node is colored a_{r-1}. This itself gives rise to two cases which we can call Case $c_{r-1,0}$ and Case $c_{r-1,r}$. More precisely, Case $c_{r,,0}$ is when in D we have all uncolored nodes above the node that is colored a_{r-1}. Case $c_{r-1,r}$ is when the node immediately above a_{r-1} is colored a_r, and everything above must be uncolored. The extra factors in these cases are:

$$\text{Case} \quad c_{r-1,0}: \quad E = \frac{(1+a_1q^N)(1+a_2q^N)...(1+a_{r-2}q^N).q^{i_1+i_2+...+i_{r-2}+i_r}}{(1-q^{i_1})(1-q^{i_2})...(1-q^{i_{r-2}})(1-q^{i_r})} \tag{5.4}$$

and

$$\text{Case} \quad c_{r-1,r}: \quad E = \frac{(1+a_1q^N)(1+a_2q^N)...(1+a_{r-2}q^N).q^{i_1+i_2+...+i_{r-2}}}{(1-q^{i_1})(1-q^{i_2})...(1-q^{i_{r-2}})}. \tag{5.5}$$

It is interesting that the extra factors in (5.4) and (5.5) when added amalgamate nicely to

$$c_{r-1,0}+c_{r-1,r}: E = \frac{(1+a_1q^N)(1+a_2q^N)...(1+a_{r-2}q^N).q^{i_1+i_2+...+i_{r-2}}.(1-q^{i_{r-1}})}{(1-q^{i_1})(1-q^{i_2})...(1-q^{i_r})}. \tag{5.6}$$

Consider now the general situation where the bottom right hand corner of D has color a_j. This situation corresponds to 2^{r-j} cases because above this node there could be colors a_{j_i}, a_{j_2}, ..., a_{j_s} where $j_1 < j_2 < .. < j_s$ could be any subset of $\{j+1, j+2, ..., r\}$. The null subset here corresponds to the case when all nodes above a_j are uncolored. In the case where the colors above a_j are specified as a_{j_1}, a_{j_2}, ..., a_{j_s}, the extra factor E is given by

$$\frac{(1+a_1q^N)(1+a_2q^N)...(1+a_{j-1}q^N)}{(1-q^{i_1})(1-q^{i_2})...(1-q^{i_{j-1}})} \cdot \frac{q^{i_1+i_2+...i_r}}{q^{i_j+i_{j_1}+i_{j_2}+...+i_{j_s}}} \cdot \frac{(1-q^{i_j})(1-q^{i_{j_1}})...(1-q^{i_{j_s}})}{(1-q^{i_j})(1-q^{i_{j+1}})...(1-q^{i_r})}$$

$$= \frac{(1+a_1q^N)(1+a_2q^N)...(1+a_{j-1}q^N)}{(1-q^{i_1})(1-q^{i_2})...(1-q^{i_r})} . q^{i_1+i_2+...+i_r} . \frac{(1-q^{i_j})}{q^{i_j}} . \prod_{t=1}^{s} \frac{(1-q^{i_{j_t}})}{q^{i_{j_t}}} . \tag{5.7}$$

Summing the expression in (5.7) over all 2^{r-j} vectors $a_{j_1}, a_{j_2}, ..., a_{j_s}$, we get

$$\frac{(1+a_1q^N)(1+a_2q^N)...(1+a_{j-1}q^N)}{(1-q^{i_1})(1-q^{i_2})...(1-q^{i_r})} . q^{i_1+i_2+...+i_r} . \prod_{t=j+1}^{r} \{1+\frac{(1-q^{i_t})}{q^{i_t}}\} . \frac{(1-q^{i_j})}{q^{i_j}}$$

$$= \frac{(1+a_1q^N)(1+a_2q^N)...(1+a_{j-1}q^N)}{(1-q^{i_1})(1-q^{i_2})...(1-q^{i_r})} . q^{i_1+i_2+...+i_{j-1}} . (1-q^{i_j}). \tag{5.8}$$

So what we have established here is that the the extra factors E corresponding to the 2^{r-j} cases corresponding to the situation where the bottom right hand node is a_j all add up and amalgamate nicely to the expression in (5.8). Thus the total number of cases including the uncolored case given in (5.2) is

$$1 + 1 + 2 + 4 + ... + 2^{r-1} = 2^r, \tag{5.9}$$

and the 2^r cases can be combined into $r+1$ groups, which correspond to the $r+1$ summands on the left in (5.9). If we now sum the expression in (5.8) over all j, we get

$$E = \frac{Z}{(1-q^{i_1})(1-q^{i_2})...(1-q^{i_r})}. \tag{5.10}$$

Finally if we attach the extra factor E in (5.10) to the common factor C in (5.1) and sum over all vectors $i_1, i_2, ...i_r$ that add up to N, and sum over all N, we get the series in (4.8) with Z in place of Y. Since we have shown that $Y = Z$, this proves the multi-dimensional identity (4.8) combinatorially.

Remarks:

(i) When $r = 2$, the grouping that we have done here is to sum the expressions in (2.12) as

$$I + II + (III + IV) \tag{5.11}$$

and not in the form $(I+III)+(II+IV)$ that we did in Section 2. In any case amalgamations can be achieved in many ways.

(ii) The grouping in (5.11) corresponds to the decomposition in (3.5) in the case $r = 2$.

In the next section we prove (4.8) q-theoretically by extending the proof in Section 3 to the r-dimensional setting. That is we give the r-dimensional version of Cayley's proof.

6. q-hypergeometric proof of the multi-dimensional identity

In view of the Lemma, and in view of our combinatorial derivation in the previous section, we will replace Y in (4.8) by Z with Z as in (4.9). We then multiply both sides of (4.8) by

$$(1+a_1)(1+a_2)...(1+a_r)$$

and denote the resulting sum $S = S(a_1, a_2, ..., a_r)$. Thus (4.8) is equivalent to

$$S =: \sum_{N\geq 0} q^{N^2} \prod_{k=1}^{r} (-a_k)_N \sum_{i_1+i_2+...+i_r=N} \frac{{a_1}^{i_1} {a_2}^{i_2} ... {a_r}^{i_r} q^{T_{i_1-1}+T_{i_2-1}+...+T_{i_r-1}} . Z}{(q)_{i_1} (q)_{i_2} ... (q)_{i_r}}$$

$$= \prod_{i=1}^{r} (-a_i)_\infty =: P, \tag{6.1}$$

where $P = P(a_1, a_2, ..., a_r)$ is defined to be the product on the right in (6.1).

The product P clearly satisfies the functional equation

$$P(a_1, a_2, ..., a_r) = (1+a_1)(1+a_2)...(1+a_r) P(a_1 q, a_2 q, ..., a_r q). \tag{6.2}$$

We will now show that the sum S satisfies the same functional equation. To this end we note that Z is a sum of $r+1$ expressions Z_j, one for each j, where j runs from 1 to $r+1$. Let us denote by S_j that part of S corresponding to the replacement of Z by Z_j in (6.1). Thus we have

$$S_{r+1} = \sum_{N\geq 0} q^{N^2} \prod_{k=1}^{r} (-a_k)_{N+1} \sum_{i_1+i_1+...i_r=N} \frac{\prod_{m=1}^{r} (a_m q)^{i_m} . \prod_{m=1}^{r} q^{T_{i_m-1}}}{(q)_{i_1} (q)_{i_2} ... (q)_{i_r}}. \tag{6.3}$$

Next

$$S_r = \sum_{N\geq 0} q^{N^2} \prod_{k=1}^{r-1} (-a_k)_{N+1} . (-a_r)_N \sum_{i_1+i_2+...+i_r=N} \frac{\prod_{m=1}^{r-1} (a_m q)^{i_m} . {a_r}^{i_r} . \prod_{m=1}^{r} q^{T_{i_m-1}}}{(q)_{i_1} (q)_{i_2} ... (q)_{i_{r-1}} (q)_{i_r-1}}. \tag{6.4}$$

Noting the change in going from (6.3) to (6.4), we see that the general case is

$$S_j = \sum_{N\geq 0} q^{N^2} \prod_{k=1}^{j-1} (-a_k)_{N+1} . \prod_{\ell=j}^{r} (-a_\ell)_N \times$$

$$\sum_{i_1+i_2+...+i_r=N} \frac{\prod_{m=1}^{j-1} (a_m q)^{i_m} . \prod_{n=j}^{r} (a_n)^{i_n} . \prod_{m=1}^{r} q^{T_{i_m-1}}}{(q)_{i_1} (q)_{i_2} ... (q)_{i_{j-1}} (q)_{i_j-1} (q)_{i_{j+1}} ... (q)_{i_r}}. \tag{6.5}$$

Proceeding in this fashion, the penultimate sum is

$$S_2 = \sum_{N\geq 0} q^{N^2}(-a_1)_{N+1}\prod_{\ell=2}^{r}(-a_\ell)_N \sum_{i_1+i_2+...+i_r=N} \frac{(a_1q)^{i_1}\cdot\prod_{n=2}^{r}(a_n)^{i_n}\cdot\prod_{m=1}^{r}q^{T_{i_m-1}}}{(q)_{i_1}(q)_{i_2-1}(q)_{i_3}...(q)_{i_r}}. \tag{6.6}$$

The final sum is

$$S_1 = \sum_{N\geq 0} q^{N^2}\prod_{\ell=1}^{r}(-a_\ell)_N \sum_{i_1+i_2+...i_r=N} \frac{\prod_{n=1}^{r}(a_n)^{i_n}\prod_{m=1}^{r}q^{T_{i_m-1}}}{(q)_{i_1-1}(q)_{i_2}...(q)_{i_r}}. \tag{6.7}$$

It is to be noted that in each of the sums S_j for $1 \leq j \leq r$, we have $i_j \geq 1$. So what we do next is to replace i_j by $i_j + 1$ and consequently N by $N + 1$ in S_j for $1 \leq j \leq r$. What this does is to make the denominators all the same for all S_j, namely

$$(q)_{i_1}(q)_{i_2}...(q)_{i_r},$$

and retain the summation condition for the indices as $i_1 + i_2 + ... + i_r = N$. But then we have the replacement

$$T_{i_1-1} + T_{i_2-1} + ... + T_{i_r-1} \mapsto i_j + T_{i_1-1} + T_{i_2-1} + ... + T_{i_r-1}.$$

Thus the sums S_j for $1 \leq j \leq r$ can be rewritten as

$$S_j = \sum_{N\geq 0} q^{(N+1)^2}\prod_{k=1}^{j-1}(-a_k)_{N+2}\cdot\prod_{\ell=j}^{r}(-a_\ell)_{N+1}\times$$

$$\sum_{i_1+i_2+...i_r=N} \frac{\prod_{m=1}^{j}(a_mq)^{i_m}.a_j\cdot\prod_{n=j+1}^{r}(a_n)^{i_n}\cdot\prod_{m=1}^{r}q^{T_{i_m-1}}}{(q)_{i_1}(q)_{i_2}...(q)_{i_r}}. \tag{6.8}$$

Note the difference between (6.5) and (6.8). Since

$$S = S_{r+1} + \sum_{j=1}^{r} S_j,$$

we see from (6.8) and (6.3) that

$$S = \prod_{k=1}^{r}(1+a_k)\sum_{N\geq 0} q^{N^2}\prod_{\ell=1}^{r}(-a_\ell)_N \sum_{i_1+i_2+...i_r=N} \frac{\prod_{m=1}^{r}(a_mq)^{i_m}\cdot\prod_{m=1}^{r}q^{T_{i_m-1}}.W}{(q)_{i_1}(q)_{i_2}...(q)_{i_r}}, \tag{6.9}$$

where

$$W = 1 + q^{2N+1}\sum_{j=1}^{r}\{a_jq^{-i_{j+1}-i_{j+2}-...-i_r}\prod_{k=1}^{j-1}(1+a_kq^{N+1})\}. \tag{6.10}$$

Finally notice that because $i_1 + i_2 + ... + i_r = N$, (6.10) can be rewritten as

$$W = 1 + \sum_{j=1}^{r} a_j q^{N+1} \prod_{k=1}^{j-1} (1 + a_k q^{N+1}).q^{\sum_{k=1}^{j} i_k} = Y_{i_1,i_2,...i_r}(a_1 q, a_2 q, ..., a_r q), \tag{6.11}$$

on comparison with (4.10). Thus from (6.9) and (6.10) and the fact that $Y = Z$, we see that S satisfies the functional equation

$$S(a_1, a_2, ..., a_r) = (1 + a_1)(1 + a_2)...(1 + a_r)S(a_1 q, a_2 q, ..., a_r q). \tag{6.12}$$

Since S and P satisfy the same initial conditions, we deduce from (6.12) and (6.2) that identity (6.1) holds and this completes the proof of the multi-dimensional Sylvester identity.

Remark: In the proof we replaced Y by Z to set up the iteration $a_i \mapsto a_i q$, but in the end we had to revert back to Y as can be seen from (6.11). Thus both versions of the multi-dimensional identity with Y and Z are needed.

7: Two problems

Even though Sylvester proved (1.1) combinatorially, he did not provide a combinatorial interpretation of it. In [1] we showed that the Sylvester identity had the following weighted partition interpretation:

Theorem A:

Let $p_d(n;k)$ denote the number of partitions of n into distinct parts, and with exactly k parts.

Let $g_3(n;\nu,\ell)$ denote the number of partitions of n of the form $b_1 + b_2 + ...b_\nu$ such that $b_i - b_{i+1} \geq 3$, of which ℓ of the gaps $b_i - b_{i+1} \geq 4$ with the convention $b_{\nu+1} = -1$. Then

$$\sum_k p_d(n;k)a^k = \sum_{\nu.\ell} g_3(n;\nu,\ell)a^\nu(1+a)^\ell.$$

It would be nice to determine a similar weighted partition implication of the multi-dimensional identity, where the series in (4.8) is interpreted as the generating function of partitions whose colored parts satisfy certain difference conditions.

Starting from Schur's theorem [9], Andrews [4],[5] produced two infinite hierarchies of partition theorems to moduli $2^r - 1$. In [2] Theorem 15, we formulated the Andrews hierarchies in terms of partitions into r primary colors and the complete alphabet of $2^r - 1$ colors generated by the r primary colors; the proof of this theorem involved the amalgamation of various generating functions in the spirit of what was demonstrated in Section 5. On seeing this, Dominique Foata asked in 1998 (after my talk at the conference in Maratea, Italy, for George Andrews' 60-th birthday) whether a key identity for the Andrews hierarchies can be constructed that extends the key identity (2.18) for Schur's theorem. However no hypergeometric *key identity* is known for the Andrews hierarchies. The second problem we would like to raise is whether our multi-dimensional identity (4.8) is the elusive key identity for the Andrews hierarchies, or whether such a key identity for the Andrews hierarchies can be constructed from ideas underlying (4.8), because in (4.8) we are using the complete alphabet of $2^r - 1$ colors generated by r primary colors. Perhaps symmetrizing the key identity (4.8) in the parameters a_1, a_2, ..., a_r might be be helpful.

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