PARTITIONS WITH NON-REPEATING ODD PARTS AND COMBINATORIAL IDENTITIES

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Abstract: *Continuing our earlier work on partitions with non-repeating odd parts and q-hypergeometric identities, we now study these partitions combinatorially by representing them in terms of 2-modular Ferrers graphs. This yields certain weighted partition identities with free parameters. By special choices of these parameters, we connect them to the Göllnitz-Gordon partitions, and combinatorially prove a modular identity and some parity results. As a consequence, we derive a shifted partition theorem mod 32 of Andrews. Finally we discuss basis partitions in connection with the 2-modular representation of partitions with non-repeating odd parts, and deduce two new parity results involving partial theta series.*

1. Introduction

Let $P_{o,d}$ denote the set of partitions in which the odd parts do not repeat. The generating function of such partitions admits the product representation

$$\prod_{m=1}^{\infty} \frac{(1+bzq^{2m-1})}{(1-zq^{2m})}, \tag{1.1}$$

where the power of b keeps track of the number of odd parts and the power of z keeps track of the total number of parts. In a recent paper [4], by representing these partitions in terms of 2-modular Ferrer's graphs, we derived the following Lebesgue type expansion for the above product:

$$\frac{(-bzq;q^2)_\infty}{(zq^2;q^2)_\infty} = \sum_{k=0}^{\infty} z^k q^{2k^2} . \frac{(-bq^{-1};q^2)_k}{(q^2;q^2)_k} . \frac{(-bzq;q^2)_k}{(zq^2;q^2)_k} . \frac{(1+bzq^{4k-1})}{(1+bzq^{2k-1})}. \tag{1.2}$$

Here and in what follows, we have made use of the standard notation

$$(a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad \text{for integers} \quad n \geq 0, \tag{1.3}$$

and

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n = \prod_{j=0}^{\infty}(1-aq^j), \quad \text{when} \quad |q|<1. \tag{1.4}$$

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For partitions in which the even parts do not repeat, there is the classic Lebesgue identity (see Andrews [8], Ch.2)

$$\frac{(-bq^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}(-bq)_k}{(q)_k}. \tag{1.5}$$

Primarily owing to this identity, partitions with non-repeating even parts have been studied extensively. Much less is known about partitions with non-repeating odd parts, but there are interesting studies due to Berkovich-Garvan [11], Hirschhorn-Sellers[16] and Radu-Sellers[19]; these partitions have also been discussed by Andrews [6], [7] in certain contexts..

To obtain a Lebesgue type expansion for the product in (1.1), we represented partitions with non-repeating odd parts by 2-modular Ferrers graphs and derived (1.2) in [4]. The goal in [4] was to connect (1.2) with several fundamental q-hypergeomteric identities whereas the emphasis here is to focus on the 2-modular representation and deduce partition identities combinatorially. The results here and in [4] are different from those discussed by the other authors and contstitute the first systematic combinatorial study of partitions with non-repeating odd parts via their 2-modular Ferrers graphs.

In the next section we connect $P_{o,d}$ surjectively with the set of partitions into parts that differ by ≥ 4 with strict inequality if a part is odd. The collection of these partitions Ψ is a subset of $P_{o,d}$. So we establish a weighted partition identity (see Theorem 2 in Section 3) in parameters b, z that connects partitions in Ψ with those in $P_{o,d}$ and explains the link between (1.2) and the surjection. Next in Section 4 we specialize the parameters in Theorem 2 and establish links with the well-known Göllnitz-Gordon partitions, deducing combinatorially a modular identity that falls out of (1.2) by these specializations. From this we also deduce in Section 4 some new parity results involving partitions with non-repeating odd parts and the Göllnitz-Gordon partitions. In Section 5 we show how the modular identity leads to a shifted partition identity mod 32 due to Andrews [9], and in Section 6 we briefly describe an alternate approach to this modular identity that we had studied earlier [2]. Finally in Section 7 we discuss basis partitions in connection with these 2-modular graphs of partitions with non-repeating odd parts, and establish new parity results using the idea of *signature* of a basis partition.

By considering variants of Ferrers graphs in different shapes, Propp [18] has shown that a number of classical partition functions as well as some new ones can be treated. In particular, he noted that partitions with non-repeating odd parts are linked to Göllnitz-Gordon partitions (with difference conditions) by considering certain special diamond shaped 2-modular diagrams. Although there are connections with Propp's work, we have exploited the connection between partitions with non-repeating odd parts and the Göllnitz-Gordon partitions to obtain parity results and the Andrews shifted partition identity. Thus our methods and results are different from Propp's.

We conclude this section with some notations and conventions.

For a partition π we let

$$\sigma(\pi) = \text{ the sum of the parts of}\,\pi\,\text{(the number being partitioned)},$$

$$\lambda(\pi) = \text{ the least part of}\,\pi, \quad \text{and} \quad \nu(\pi) = \text{ the number of parts of }\,\pi.$$

If we count the number of parts of a specific type, we denote that with a subscript. For example, $\nu_o(\pi)$ (resp $\nu_e(\pi)$) will denote the number of odd (resp. even) parts of π.

In what follows we will discuss only the 2-modular Ferrers graphs of partitions with non-repeating odd parts. When we refer to the Durfee square, we mean the largest square of nodes in such 2-modular graphs.

Finally, with regard to the q-product notation in (1.3) and (1.4), for simplicity we will write $(a)_n$ in place of $(a;q)_n$ such as in (1.5) when the base is q, but when the base is anything other than q, it will be displayed such as in (1.2). Further notations and conventions will be explained as we go along.

2. Hooks, special partitions, and primary partitions

The 2-modular Ferrers graph of a partition $\pi = b_1 + b_2 + \cdots + b_\nu$ is a left justified graph with the i-th row representing the part b_i as a row of twos and ending in a one precisely when b_i is odd. In [4] we studied partitions with non-repeating odd parts in terms of their 2-modular Ferrers graphs because in such graphs the ones (if any) will occur in corners. Thus by reading the graph columnwise, we get the 2-modular graph of another partition $\pi^* \in P_{o,d}$. Also π and π^* will have the same number of odd parts. It was this convenient property that enabled us to obtain the series representation in (1.2). With regard to the Durfee square of such Ferrers graphs, the bottom right hand corner could have a one or a two, but the rest of the entries in the Durfee squares will all be twos.

Terminology: In a two modular Ferrers graph, by an odd (resp. even) row (resp. column), we mean a row (resp. column) that ends in a one (resp. two). By the *length* of a row or a column, we mean the number of nodes in it. By the *size* of a row or column, we mean the sum of the numbers at its nodes, namely the integer that it represents.

Note that in (1.2), the last factor in $(-bzq;q^2)_k$ is $(1+bzq^{2k-1})$ which is also in the denominator. The first factor of $(-bq^{-1};q^2)_k$ is $(b+q)/q$. If we combine these observations along with the decomposition

$$(b+q)(1+bzq^{4k-1}) = q(1+bzq^{2k-1})(1+bq^{2k-1}) + b(1-zq^{2k})(1-q^{2k}), \tag{2.1}$$

then (1.2) can be rewritten as

$$\frac{(-bzq;q^2)_\infty}{(zq^2;q^2)_\infty} = \sum_{k=0}^{\infty} z^k q^{2k^2} . \frac{(-bq;q^2)_k}{(q^2;q^2)_k} . \frac{(-bzq;q^2)_k}{(zq^2;q^2)_k}$$

$$+\sum_{k=1}^{\infty} bz^k q^{2k^2-1} . \frac{(-bq;q^2)_{k-1}}{(q^2;q^2)_{k-1}} . \frac{(-bzq;q^2)_{k-1}}{(zq^2;q^2)_{k-1}} =: \Sigma_0 + \Sigma_1. \tag{2.2}$$

If the 2-modular Ferrers graphs of partitions $\pi \in P_{o,d}$ are classified according to their Durfee squares, then Σ_0 is the generating function of such Ferrers graphs in which the Durfee square has twos everywhere, whereas Σ_1 in (2.2) is the generating function of Ferrers graphs which have a one in the right hand bottom corner of the Durfee square and twos everywhere else in the Durfee square. Indeed it was (2.2) that was derived first in [4], and from this the smoother form (1.1) was deduced using (2.1). We now will discuss the partition interpretation of the terms in (2.2).

In Σ_0, the term

$$\frac{q^{2k^2}}{(q^2;q^2)_k} \tag{2.3}$$

can be thought of as of given by a $k \times k$ Durfee square of twos with columns of twos of length$\leq k$ placed to the right of the Durfee square; If in such a graph we add the numbers at the nodes along the hooks, we see that the term in (2.3) is the generating function of partitions into even parts that differ by ≥ 4. When the term in (2.3) is multiplied by factor $(-bzq;q^2)_k$, we may interpret this as inserting at most one column of length i, for each i between 1 and k, each such column having a 1 at the bottom and twos everywhere. If we now add the numbers at the nodes along the hooks of this graph, we get some odd parts whenever a column ending in a one is inserted. This yields odd parts, but then the gap is > 4 if we add numbers along the hooks.

This leads us to consider the set Ψ of partitions into parts that differ by ≥ 4 with strict inequality when a part is odd. Let us call such partitions *special*. What we have noted above is that

$$z^k q^{2k^2} \frac{(-bq;q^2)_k}{(q^2;q^2)_k} \tag{2.4}$$

is the generating function of special partitions $\overline{\pi}$ into k parts with the power of b keeping track of $\nu_o(\overline{\pi})$, and with smallest part ≥ 2. Similarly

$$bz^k q^{2k^2-1} \frac{(-bq;q^2)_{k-1}}{(q^2;q^2)_{k-1}} \tag{2.5}$$

is the generating function of special partitions into k parts with smallest part 1, with the power of b keeping track of the number of odd parts.

At this stage it is convenient to introduce some further notation and terminology. If we represent a partition π by a 2-modular graph, the portion below the Durfee square represents a partition which we denote by π_b, and the portion to the right of the Durfee square represents another partition which we denote by π_r. Typically, we discuss the conjugate π_r^* instead of π_r.

Very important in our discussion below are partitions π for which π_b is empty. We call such a partition *primary* and we let P denote the set of primary partitions within $P_{o,d}$.

If we represent a partition π by a 2-modular Ferrers graph, we could form a new partition $\rho(\pi) = \overline{\pi}$ from π by adding the numbers along the hooks of the graph. Our discussion above can be put in the form of a

Lemma 1: *The hook operation or mapping*

$$\pi \mapsto \rho(\pi) = \overline{\pi}$$

is a surjection from $P_{o,d}$ to Ψ, and a <u>bijection</u> between P and *Ψ.*

Note that both P and Ψ are subsets of $P_{o,d}$. Given $\overline{\pi} \in \Psi$, it is of interest to determine the size of the inverse image $\rho^{-1}(\overline{\pi})$. This is the same as asking how we may construct all the partitions in $P_{o,d}$ from the subset of primary partitions. We shall answer this question in the next section and determine the weights that need to be attached to each $\pi \in P$ so that when one sums these weights over the primary partitions of n, one gets $P_{o,d}(n)$, the number of partitions of n with non-repeating odd parts.

In (2.4) and (2.5) we only interpreted a part of the terms in terms of special partitions or primary partitions. The collection of all partitions in $P_{o,d}$ can be obtained from the primary partitions by means of a *sliding operation* as will be shown in the next section. This construction of all partitions of $P_{o,d}$ using the sliding operation will explain the role of the factors

$$\frac{(-bzq;q^2)_k}{(zq^2;q^2)_k} \quad \text{and} \quad \frac{(-bzq;q^2)_{k-1}}{(zq^2;q^2)_{k-1}}$$

in (2.2).

3. The sliding operation and a weighted partition theorem

Recall that primary partitions have nothing below the Durfee square. By a *sliding operation* on a partition π, we mean the removal of certain columns from π_r, and the placement of these columns as rows below the Durfee square. The collection of all unrestricted partitions of an integer can be obtained from primary partitions of that integer by performing all possible sliding operations and we studied this aspect closely in [3]. In the case of partitions in $P_{o,d}$, one has to pay attention to how these sliding operations are performed and that is what we discuss now. Before that, we note:

Under a sliding operation on a partition π, *the hook sizes are invariant*, and so $\rho(\pi) = \overline{\pi}$ is invariant. This is crucial in our arguments below.

Given a primary partition π, suppose there are ℓ odd columns in π_r. Then any of these columns could be slid down and placed as a row below the Durfee square. So for each of these odd columns we have two choices - we may slide or not slide. Thus there are 2^ℓ choices.

With regard to the even columns, suppose there are n_i columns of twos of length i. Then we could slide down 0, or 1, or 2, ..., or all n_i columns, thereby providing $n_i + 1$ choices. Now sliding the odd columns and the even columns of different lengths are "independent". Thus each primary partition π will spawn

$$\omega(\pi) = 2^{\nu_o(\pi_r^*)} \prod_i (n_i + 1) \tag{3.1}$$

partitions in $P_{o,d}$, and this would be the weight attached to π. However it is to be noted the partitions in $P_{o,d}$ obtained in this fashion have the property that in the Ferrers graph, no odd row below the Durfee square is equal to any odd column to the right of the Durfee square. We call this Case 1.

Next we discuss Case 2, which involves partitions in $P_{o,d}$ where some odd row below the Durfee square could equal an odd column to the right of the Durfee square. In order to determine this we define a *junction* in the 2-modular graph of a primary partition to be a 2 occupying a corner position such that the part in the next row above is at least larger by two. So if we have a junction, we could split the 2 in the corner as a 1+1 and create two equal odd columns whose lengths will be different from any of the odd columns to the right of the Durfee square. We then slide one of these two equal odd columns and place it as a row below the Durfee square. We can perform this splitting operation at every junction. The important thing is that ALL partitions coming under Case 2 can be obtained in this manner by performing the splitting operation at the various junctions, sliding one of each resulting pair of odds, and sliding the even columns arbitrarily. Now the splitting of a two at each junction will affect the number of columns of evens both to the right and to the left of the junction. So we have to be careful, and for this purpose we define a *secondary* partition π' generated by a primary partition π to be one in which the splitting is done at a certain number of junctions, one odd column is slid down from each split junction, but NO other columns to the right of the Durfee square are slid down.

Next denote by S_π the set of secondary partitions generated by a primary partition π. Given $\pi' \in S_\pi$, it will still have $\ell = \nu_o(\pi_r^*)$ columns to the right of the Durfee square which are not equal to any of the rows below the Durfee square of π', because the rows of π'_b came out of splitting only the even rows ending in junctions. Suppose the Ferrers graph of π' has n'_i columns of twos of length i. These columns of twos could be slid down as before. Thus each secondary partition π' spawns

$$\omega(\pi') = 2^{\nu_o(\pi_r^*)} \prod_i (n'_i + 1) \tag{3.2}$$

partitions covered under Case 2. Thus each primary partition will spawn

$$w(\pi) = \omega(\pi) + \sum_{\pi' \in S_\pi} \omega(\pi') \tag{3.3}$$

partitions in $P_{o,d}$.

Remark We may merge (3.1) into (3.2) by observing that the expression in (3.1) corresponds to the case of (3.2) where <u>no splitting</u> is done at any junction. But we have preferred to treat the cases of split junctions and the unsplit case separately. In any case, the weight $w(\pi)$ we have determined yields

Theorem 1: *Let $P_{o,d}(n)$ denote the number of partitions of n with non-repeating odd parts. Then*

$$P_{o,d}(n) = \sum_{\pi \in P, \sigma(\pi)=n} w(\pi).$$

The weight formula in (3.3) may look complicated. But what we can do is to evaluate these weights as functions of parameters b, z, and then specialize these parameters to get a significant collapse and simplification. More precisely, we first note that $\omega(\pi)$ in (3.1) can be replaced by

$$(3.4) \qquad \omega(\pi; b, z) = b^{\nu_o(\pi)} z^{\nu(\pi)} (1+z)^{\nu_o(\pi_r^*)} \prod_i (1 + z + z^2 + ... + z^{n_i}).$$

Similarly, if a secondary partition is born out of a primary partition π by splitting h out of the j junctions, then $\omega(\pi')$ in (3.2) can be replaced by

$$(3.5) \qquad \omega(\pi'; b, z) = b^{2h} b^{\nu_o(\pi)} z^h z^{\nu(\pi)} (1+z)^{\nu_o^*(\pi_r)} \prod_i (1 + z + z^2 + ... + z^{n_i'}).$$

Finally (3.3) is replaced by

$$(3.6) \qquad w(\pi; b, z) = \omega(\pi; b, z) + \sum_{\pi' \in S_\pi} \omega(\pi').$$

Thus Theorem 1 can be refined to:

Theorem 2: *The partitions in $P_{o,d}$ and the primary partitions are related by the weighted identity*

$$\sum_{\pi \in P_{o,d}, \sigma(\pi)=n} b^{\nu_o(\pi)} z^{\nu(\pi)} = \sum_{\pi \in P; \sigma(\pi)=n} w(\pi; b, z)$$

In deriving Theorems 1 and 2, used the set S_π of secondary partitions generated by a primary partition π, but did not say anything about the size of this set. We determine this now since it is of intrinsic interest.

If a primary partition π has $j = j(\pi)$ junctions, then for each junction we have choice of either to split it or not to split it. Thus each primary partition generates $\leq 2^j - 1$ secondary partitions, because if we do not split any of the junctions, that would correspond to the primary partition itself. Thus

$$|S_\pi| \leq 2^{j(\pi)} - 1. \tag{3.7}$$

The reason that there is an inequality in (3.7) is because if there two consecutive junctions such that the sizes of their rows differ by exactly 2 (equivalently in the partition $\overline{\pi}$ generated by π, the two corresponding parts differ by exactly 6), then it is not possible to split them simultaneously. Thus given the set of junctions of π, we need to consider chains of junctions in it, where by a chain χ, we mean a maximal sequence of consecutive junctions such that their corresponding rows differ by exactly 2. Thus the set of junctions is a union of these chains. Note that if a chain has only one element in it, then we have complete freedom in choosing whether to split it or not. We call such a junction as an *independent junction*. In contrast two junctions will be called *dependent* if their row sizes differ by exactly 2.

Now given a chain χ whose length (= number of junctions in it) $\ell(\chi) \geq 2$, the number of ways of splitting is the number of ways of choosing a subset of junctions no two of which are dependent (consecutive). The number of such choices is given by

Lemma 2: *The number of subsets of the first n positive integers such that these subsets can never have a pair of consecutive integers, is F_{n+2}, where $\{F_m\}$ is the Fibonacci sequence given by $F_0 = 0, F_1 = 1$, and $F_m = F_{m-1} + F_{m-2}$, for $m \geq 2$.*

The lemma is well known and is easily proved by induction on m.

From the Lemma it follows that if $\ell(\chi) \geq 2$, then the number of ways of splitting these junctions is $F_{\ell(\chi)+2}$. Note that if $\ell(\chi) = 1$, that is if the junction is independent, then the number of choices is $2 = F_3 = F_{\ell(\chi)+2}$ even in this case. Thus we have

Theorem 3: *Given a primary partition π, the number of secondary partitions generated by π is given by*

$$|S(\pi)| = \{\prod_{\chi} F_{\ell(\chi)+2}\} - 1,$$

where the product is taken over all chains of junctions including chains of length 1, and 1 is subtracted since the primary partition π is not in S_π.

Remarks:

(i) We have given in Theorem 2 the weights to be attached to the primary partitions. If the primary partitions are converted to special partitions by hooks, then these weights will be given by difference conditions on the parts. We note that Theorem 2 has been proved combinatorially and is the partition theoretic version of, and equivalent to, (2.2).

(ii) It is to be noted, that once the secondary partitions have been determined, the weights to be assigned to each secondary partition is by the same rule (3.5).

In the next section we will make special choices of the parameters b and z, to induce significant cancellation owing to the collapse of the weights. This will lead to links with the Göllnitz-Gordon partitions, and yield some nice parity results as well.

4. Göllnitz-Gordon partitions and parity results

The famous Göllnitz-Gordon identities are

$$G(q) =: \sum_{k=0}^{\infty} \frac{q^{k^2}(-q;q^2)_k}{(q^2;q^2)_k} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}, \tag{4.1}$$

and

$$H(q) =: \sum_{k=0}^{\infty} \frac{q^{k^2+2k}(-q;q^2)_k}{(q^2;q^2)_k} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{4.2}$$

These identities are the perfect analogues to the modulus 8 for what the celebrated Rogers-Ramanujan identities are to the modulus 5. The identities were discovered independently by Göllnitz [12] and Gordon [13]. The partition interpretation of the products in (4.1) and (4.2) is obvious - they are the generating functions of partitions into parts $\equiv 4, \pm 2i-1 \pmod 8$, for $i = 1, 2$. The partition interpretation of the series in (4.1) and (4.2) is that for $i = 1, 2$, the k-th terms are the generating functions of partitions into k parts that differ by at least 2, with strict inequality if a part is even, and with least part $\geq 2i-1$. We call the partitions satisfying these difference conditions as Göllnitz-Gordon partitions of the first and second kind and denote the set of such partitions by GG_1 and GG_2 respectively.

We now set the free parameter $z = -1$ and see what the effect is combinatorially. Given a primary partition π and all its secondaries, consider the set of all partitions generated by performing sliding operations. Note that in the weight formula $w(\pi; b, z)$ in (3.6), there always a factor $(1+z)$ if $\nu_o(\pi_r^*) \geq 1$, and this factor would make the weight 0 when $z = -1$. Thus for the weight to be non-zero, we must have $\nu_o(\pi_r^*) = 0$. This means that except for the case where the bottom right node in the Durfee square is a 1, all other nodes in the graph must be twos, other than the pairs of odds generated by the splitting of junctions. Let us for the moment not consider the the pairs of odd rows obtained by splitting the junctions but focus on the remainder of the graph in the case when the bottom right entry in the Durfee square is a 2. Note at this stage that the sums

$$1 + z + z^2 + ... + z^{n_i} \quad \text{and} \quad 1 + z + z^2 + ... + z^{n_i'}$$

become zero if n_i or n_i' is odd, and equal to 1 when n_i or n_i' is even. So the non-zero contribution to $w(\pi; b, -1)$ comes from those π or $\pi' \in S_\pi$ whose columns of twos will occur in pairs. Denote by π'' the subgraph of π or π' by ignoring (for the moment) the pairs of odd columns from split junctions. The partition $\rho(\pi'')$ generated by hooks of π'' will be a partition into even parts differing by ≥ 4 with all parts $\equiv 2(mod\, 4)$. If we now consider the columns ending in junctions, the effect on the hook size is to create parts that are $\equiv 0(mod\, 4)$, but then the gap is > 4. Thus we have Göllnitz-Gordon partitions $\rho(\pi)$ of type 1, dilated by a factor of 2, and counted with weight $(-1)^{\nu(\overline{\pi})} b^{2h}$. This means if we take $b = 1$, the generating function is $G(-q^2)$, with G defined by (4.1). The discussion we have presented now is for the case where the Durfee square of π has a 2 at the bottom right. By similar reasoning, it can be shown that when there is a 1 at the bottom right corner of the Durfee square, then with the choice $z = -1$, the only surviving graphs are those for which the hook sizes (except for the isolated 1 at the bottom right of the square) are Göllnitz-Gordon partitions of type 2 dilated by a factor of 2, and counted with weight $(-1)^{\nu(\overline{\pi})-1} b^{2h}$ Thus if we take $b = 1$, the generating function in this case is $-qH(-q^2)$. So what we have given here is a combinatorial derivation of the modular identity

$$\frac{(q; q^2)_\infty}{(-q^2; q^2)_\infty} = G(-q^2) - qH(-q^2). \tag{4.3}$$

Remarks: (i) By setting $z = -1$, $b = 1$ in (2.2), we get

$$\frac{(q; q^2)_\infty}{(-q^2; q^2)_\infty} = \sum_{k=0}^{\infty} (-1)^k q^{2k^2} . \frac{(q^2; q^4)_k}{(q^4; q^4)_k} - q \sum_{k=0}^{\infty} q^{2k^2+4k} . \frac{(q^2; q^4)_k}{(q^4; q^4)_k}$$

$$= G(-q^2) - qH(-q^2) \tag{4.4}$$

which is an analytic derivation of the modular identity from (2.2). Note that in (4.4) both sums have been made to start at $k = 0$ to connect with $H(-q^2)$, whereas in (2.2), Σ_1 starts at $k = 1$. What is important here is that (4.3) can be derived purely combinatorially by discussing the collapse of the weights under the sliding operation.

(ii) We need only $z = -1$ to induce a collapse of the weights, and can keep b as a feee parameter. We chose $b = 1$ only to connect with the Göllnitz-Gordon functions $G(q)$ and $H(q)$ as they were originally defined. If we keep b as a free parameter when $z = -1$, then (2.2) simplifies to

$$\frac{(bq;q^2)_\infty}{(-q^2;q^2)_\infty} = \sum_{k=0}^{\infty}(-1)^k q^{2k^2}.\frac{(b^2q^2;q^4)_k}{(q^4;q^4)_k} - bq\sum_{k=0}^{\infty}(-1)^k q^{2k^2+4k}.\frac{(b^2q^2;q^4)_k}{(q^4;q^4)_k}$$

$$=: G(-q^2;b^2) - bqH(-q^2;b^2), \tag{4.5}$$

The combinatorial arguments given above prove not only (4.4), but also (4.5).

(iii) The expression on the left hand side of (4.5) is a function of b and q. The right hand side simultaneously provides a decomposition of this function as an even function of both b and q and an odd function of b and q. This bisection into the even and odd components yields the parity theorems stated in this section connecting partitions in $P_{o,d}$ with the Göllnitz-Gordon partitions. Next in Section 5, by using the Göllnitz-Gordon identities in (4.4), we will deduce an interesting shifted partition theorem of Andrews [9]. Finally in Section 6 we provide yet another approach to the general modular identity (4.5) and present a finite version of it as well.

Parity results

By comparing the coefficients of q^{2n} on both sides of (4.5) we get

Theorem 4:

$$\sum_{\pi\in P_{o,d},\sigma(\pi)=2n}(-1)^{\nu(\pi)}b^{\nu_o(\pi)} = (-1)^n\sum_{\overline{\pi}\in GG_1,\sigma(\overline{\pi})=n}b^{2\nu_e(\overline{\pi})}$$

Similarly, by comparing coefficients of q^{2n+1} on both sides of (4.5) we deduce

Theorem 5:

$$\sum_{\pi\in Po,d,\sigma(\pi)=2n+1}(-1)^{\nu(\pi)}b^{\nu_o(\pi)} = -b(-1)^n\sum_{\overline{\pi}\in GG_2,\sigma(\overline{\pi})=n}b^{2\nu_e(\overline{\pi})}$$

Let $gg_1(n)$ and $gg_2(n)$ denote the number of Göllnitz-Gordon partitions of type 1 and 2 respectively. Also let $p_e(n)$ (resp. $p_o(n)$) denote the number of partitions $\pi \in P_{o,d}$ of n with an even (resp. odd) number of parts. Then with $b = 1$ in Theorems 4 and 5, we get

Corollary 1:

$$p_e(2n) - p_o(2n) = (-1)^n gg_1(n)$$

Corollary 2:

$$p_e(2n+1) - p_o(2n+1) = (-1)^{n+1} gg_2(n)$$

Corollaries 1 and 2 yield

Corollary 3:

$$(i) \quad p_e(m) \geq p_o(m), \quad \text{if} \quad m \equiv 0, 3 (mod\, 4)$$

$$(ii) \quad p_o(m) \geq p_e(m), \quad \text{if} \quad m \equiv 1, 2 (mod\, 4)$$

Remarks: (i) In the case of partitions into distinct parts, the parity split of such partitions based on the number of parts, yields the pentagonal series. This is to be compared with the parity split of partitions in which the even parts do not repeat; by setting $b = -1$ in Lebesgue's identity (1.5) we get Gauss' triangular numbers theorem which is on par with Euler's pentagonal numbers theorem. In contrast, the parity split of unrestricted partitions of an integer does not lead to a lacunary series; instead we get partitions into distinct odd parts. This is to be compared with the parity split of partitions with non-repeating odd parts that we have discussed here, where we get the Göllnitz-Gordon partitions instead of a lacunary series.

(ii) Note that in Corollaries 1 and 2 the parity split is on $P_{o,d}$, and so one may ask whether there is a result involving the parity split of the Göllnitz-Gordon partitions themselves. The answer is yes, and this is obtained by choosing $b = \sqrt{-1}$ in Theorems 4 and 5. Interpreting that combinatorially we get the following:

Theorem 6: *When $m = 2n$, and $\sigma(\pi) = m$, we have $\nu_o(\pi)$ is even, so*

$$\sum_{\pi \in P_{o,d}, \sigma(\pi)=2n, \nu_o(\pi) \equiv 0 (mod\, 4)} (-1)^{\nu(\pi)} \quad - \quad \sum_{\pi \in P_{o,d}, \sigma(\pi)=2n, \nu_o(\pi) \equiv 2 (mod\, 4)} (-1)^{\nu(\pi)}$$

$$= (-1)^n \sum_{\overline{\pi} \in GG_1, \sigma(\overline{\pi})=n} (-1)^{\nu_e(\overline{\pi})}$$

Theorem 7: *When $m = 2n+1$, and $\sigma(\pi) = m$, we have $\nu_o(\pi)$ is odd, so*

$$\sum_{\pi\in P_{o,d},\sigma(\pi)=2n+1,\nu_o(\pi)\equiv 1(mod\,4)} (-1)^{\nu(\pi)} - \sum_{\pi\in P_{o,d},\sigma(\pi)=2n+1,\nu_o(\pi)\equiv 3(mod\,4)} (-1)^{\nu(\pi)}$$

$$(-1)^{n+1} \sum_{\overline{\pi}\in GG_2,\sigma(\overline{\pi})=n} (-1)^{\nu_e(\overline{\pi})}$$

5. A shifted partition identity modulo 32 of Andrews

If we take the product forms of $G(-q^2)$ and $H(-q^2)$ given by (4.1) and (4.2), and substitute them in (4.4), we get

$$\frac{(q;q^2)_\infty}{(-q^2;q^2)_\infty}$$

$$= \frac{1}{(-q^2;q^{16})_\infty(q^8;q^{16})_\infty(-q^{14};q^{16})_\infty} - \frac{q}{(-q^6;q^{16})_\infty(q^8;q^{16})_\infty(-q^{10};q^{16})_\infty}. \tag{5.1}$$

This can be rewritten as

$$1 = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^{16})_\infty(q^8;q^{16})_\infty(-q^{14};q^{16})_\infty}$$

$$- \frac{q(-q^2;q^2)_\infty}{(q;q^2)_\infty(-q^6;q^{16})_\infty(q^8;q^{16})_\infty(-q^{10};q^{16})_\infty}. \tag{5.2}$$

At this stage we use Euler's trick to replace $(-q^2;q^2)_\infty$ in the numerator of (5.2) by $(q^2;q^4)_\infty$ in the denominator to get

$$1 = \frac{1}{(q;q^2)_\infty(q^2;q^4)_\infty(-q^2;q^{16})_\infty(q^8;q^{16})_\infty(-q^{14};q^{16})_\infty}$$

$$- \frac{q}{(q;q^2)_\infty(q^2;q^4)_\infty(-q^6;q^{16})_\infty(q^8;q^{16})_\infty(-q^{10};q^{16})_\infty}$$

$$= \prod_{j>0\, odd\, or\, j\equiv\pm4,\pm6,\pm8,\pm10(mod\,32)} \frac{1}{(1-q^j)}$$

$$- q \prod_{j>0,\, odd\, or\, j\equiv\pm2,\pm8,\pm12,\pm14(mod\,32)} \frac{1}{(1-q^j)}. \tag{5.3}$$

We now compare the coefficients of q^n on both sides of (5.3) and note the coefficient is 0 for $n \geq 1$. This yields the following shifted partition theorem of Andrews [9]:

Theorem A:

Let $p_S(n)$ denote the number of partitions of n with parts coming from the set S given by $S = \{j \in Z | j > 0\, is\, odd\, or\, j \equiv \pm 4, \pm 6, \pm 8, \pm 10 (mod\, 32)\}$.

Let $p_T(n)$ denote the number of partitions of n with parts coming from the set T given by $T = \{j \in Z | j > 0\, is\, odd\, or\, j \equiv \pm 2, \pm 8, \pm 12, \pm 14 (mod\, 32)\}$. Then

$$p_S(n) = p_T(n-1).$$

Remarks:

(i) This was the first shifted partition identity to be stated in the literature. It was Andrews' contribution to the American Mathematical Monthly for the Ramanujan Centennial.

(ii) It is to be noted that only our proof of the modular relation (4.4) was combinatorial. From there we used the product form of the Göllnitz-Gordon identities to arrive at (5.3). So our proof of Theorem A is only partly combinatorial but different from Andrews' proof.

(iii) In [2], a modular relation close to (4.4) for the Göllnitz-Gordon functions was used to give a new proof of the Göllnitz-Gordon identities. In view of this, we will present in the next section a different and direct proof of the general modular identity (4.5).

6. Alternate approach to the modular identity with a parameter

We will now discuss an alternate approach to the modular identity (4.5) using ideas in our earlier work [1].

A simple direct way to determine the even and odd parts of the expression on the left hand side of (4.5) is to expand $(bq; q^2)_\infty$ as an infinite series, and extract the even and odd parts from this. More precisely, start with

$$\frac{(bq; q^2)_\infty}{(-q^2; q^2)_\infty} = \frac{1}{(-q^2; q^2)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k b^k q^{k^2}}{(q^2; q^2)_k}. \tag{6.1}$$

It is clear from the expansion in (6.1) that the even and odd parts are given by setting the index k to be even and odd. That is

$$\frac{(bq;q^2)_\infty}{(-q^2;q^2)_\infty} = \frac{1}{(-q^2;q^2)_\infty}\{\sum_{k=0}^{\infty}\frac{b^{2k}q^{4k^2}}{(q^2;q^2)_{2k}} - \sum_{k=0}^{\infty}\frac{b^{2k+1}q^{4k^2+4k+1}}{(q^2;q^2)_{2k+1}}\} \tag{6.2}$$

provides a simultaneous decomposition into even and odd functions of both b and z. From (6.2) we see that proving (4.5) is equivalent to showing

$$\frac{1}{(-q^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{b^{2k}q^{4k^2}}{(q^2;q^2)_{2k}} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell q^{2\ell^2}(b^2q^2;q^4)_\ell}{(q^4;q^4)_\ell}. \tag{6.3}$$

and

$$\frac{1}{(-q^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{b^{2k}q^{4k^2+4k}}{(q^2;q^2)_{2k+1}} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell q^{2\ell^2+4\ell}(b^2q^2;q^4)_\ell}{(q^4;q^4)_\ell}. \tag{6.4}$$

If in (6.3) and (6.4) we replace the $(-q^2;q^2)_\infty$ in the denominator on the left by $(q^2;q^4)_\infty$ in the numerator, and then move it inside the summation to absorb certain factors $(q^2;q^4)_k$ in the denominator, then (6.3) and (6.4) can be rewritten as

$$\sum_{k=0}^{\infty}\frac{b^{2k}q^{4k^2}(q^{4k+2};q^4)_\infty}{(q^4;q^4)_k} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell q^{2\ell^2}(b^2q^2;q^4)_\ell}{(q^4;q^4)_\ell}, \tag{6.5}$$

and

$$\sum_{k=0}^{\infty}\frac{b^{2k}q^{4k^2+4k}(q^{4k+6};q^4)_\infty}{(q^4;q^4)_k} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell q^{2\ell^2+4\ell}(b^2q^2;q^4)_\ell}{(q^4;q^4)_\ell}. \tag{6.6}$$

Identities (6.5) and (6.6) can be proved easily by expanding the factor $(b^2q^2;q^4)_\ell$ in the numerator on the right by the q-binomial theorem to get a double series, and reversing the order of summation to get the series on the left. We note that (6.5) and (6.6) are consequences of the following Lemma that was used in [1] for certain applicatons:

Transformation Lemma:

$$\sum_{k=0}^{\infty}\frac{(ab)^k q^{2k^2}(-aq^{2k+1};q^2)_\infty}{(q^2;q^2)_k} = \sum_{\ell=0}^{\infty}\frac{a^\ell q^{\ell^2}(-bq;q^2)_\ell}{(q^2;q^2)_\ell}$$

Remarks:

(i) The series on the right in the Lemma is a two parameter refinement of the Göllnitz-Gordon function $G(q)$. Owing to the presence of the second parameter a, the replacement $a \mapsto aq^2$ gives a two parameter refinement of the second Göllnitz-Gordon function $H(q)$, and the replacement $a \mapsto aq$ yields a refinement of the Second Little Göllnitz Series. Similarly $a \mapsto aq$, $b \mapsto bq^{-2}$ yields the First Göllnitz-Gordon Series. The Little Göllnitz-Gordon identities are dilated versions of the Lebesgue identity (1.5).

(ii) Identity (6.5) is a special case of the Lemma with the choices

$$q \mapsto -q^2, \quad a = 1, \quad \text{and} \quad b \mapsto b^2.$$

Identity (6.6) follows from the Duality Lemma by the choices

$$q \mapsto -q^2, \quad a = q^4 \quad \text{and} \quad b \mapsto b^2.$$

(iii) We close this section by pointing out that there is an interesting finite version of the Transformation Lemma which can be easily proved either q-theoretically or combinatorially:

Finite Transformation Lemma: *For arbitrary integers m, n, we have*

$$\sum_{k=0}^{m} (ab)^k q^{2k^2} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (-aq^{2k+1}; q^2)_n = \sum_{\ell=0}^{m+n} a^\ell q^{\ell^2} \sum_{k=0}^{\ell} b^k q^{k^2} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n \\ \ell - k \end{bmatrix}_{q^2}.$$

The Transformation Lemma follows from the Finite Transformation Lemma by letting $m, n \to \infty$. In view of Remark (i) above, a finite version of Lebesgue's identity can be obtained by replacing $a \mapsto aq$, $b \mapsto bq$ and $q^2 \mapsto q$ in that order in the Finite Transformation Lemma.

7. Basis partitions in $P_{o,d}$ and their signature

Our goal here is to describe some new results on basis partitions for partitions with non-repeating odd parts, but we need to give as background and motivation, certain results on basis partitions within the set of unrestricted partitions.

Let an unrestricted partition $\pi; b_1 + b_2 + ... + b_\nu$ be represented as an ordinary Ferrers graph, and let $\pi^* : c_1 + c_2 + ... + c_{b_1}$ be its conjugate. If this graph has a $k \times k$ Durfee square, then its successive rank vector is given by $\mathbf{r}$: $(r_1, r_2, ..., r_k)$, where $r_i = b_i - c_i$, for $1 \leq i \leq k$. Given a successive rank vector **r**, consider the partition π which has **r** as its successive rank vector, and with $\sigma(\pi)$ minimal. Such a partition is called a *basis partition* and they were first considered by Gupta [14]. Subsequently, Nolan, Savage and Wilf [17] showed that if $b(n)$ denotes the number of basis partitions of n, then the generating function is

$$\sum_{n=0}^{\infty} b(n)q^n = \sum_{k=0}^{\infty} \frac{q^{k^2}(-q)_k}{(q)_k}. \tag{7.1}$$

Hirschhorn [15] then interpreted (7.1) as a weighted partition identity connecting $b(n)$ with the Rogers-Ramanujan partitions of n with weights as powers of 2.

Basis partitions are characterized by the property that no row below the Durfee square is equal to any column to the right of the Durfee square in the ordinary Ferrers graph. Thus there is no redundancy in a basis partition. Owing to this property, we noticed in 2005 (see [5]) that

$$\sum_{k=0}^{\infty} \frac{q^{k^2}(-zq)_k}{(q)_k} =: \sum_{n,j} b(n;j)z^j q^n, \tag{7.2}$$

is the generating function of $b(n;j)$, the number of basis partitions of n with *signature* j, where by the signature of a basis partition π, we mean the number of different parts in π_b, the portion of π below its Durfee square. This observation on the signature immediately yields the following parity result as a consequence of (7.2) as noted in [5]:

Theorem 8: *Let* $b_e(n)$ *(resp.* $b_o(n)$*) denote the number of basis partitions of* n *with even (resp. odd) signature. Then*

$$b_e(n) - b_o(n) = 1, \quad \text{if} \quad n \quad \text{is a perfect square, and 0 otherwise.}$$

In addition, the signature enables us [5] to refine Hirschhorn's weighted partition theorem by replacing his powers of 2 weights by powers of $(1 + z)$ (see Theorem 4 of [5]).

In [5] we interpreted (7.2) in terms of the signature because we constructed basis partitions from Rogers-Ramanujan partitions by means of the sliding operation. Recently Andrews [10] has obtained an analytic expression for the series in (7.2) for all z which yields the first Rogers-Ramanujan identity when $z = 0$. We now describe some new results on basis partitions defined in terms of the 2-modular Ferrers graphs of partitions in $P_{o,d}$. As we stressed at the beginning, by Ferrers graphs we mean only such 2-modular graphs. That is why in this section we referred to the graphs of unrestricted partitions as *ordinary* Ferrers graphs, which are not 2-modular graphs.

Given a 2-modular Ferrers graph of $\pi \in P_{o,d}$ with a $k \times k$ Durfee square, we define its successive rank vector to be $\mathbf{r}$=$(r_1, r_2, ..., r_k)$, where

$$r_i = \text{(size of the i-th row)} \quad - \quad \text{(size of the i-th column)}. \tag{7.3}$$

The size of a row (resp. column) was defined in Section 2 as the sum of the entries at the nodes in that row (resp. column), in contrast to the *length* of a row or column which is simply the number of nodes in it*. By a *minimal basis partition* $\mu \in P_{o,d}$, we mean a partition for which $\sigma(\mu)$ is minimal with respect to a given successive rank vector. It is to be noted that any vector $(r_1, r_2, ..., r_k)$ of integers can occur as a successive rank vector $\mathbf{r}$ of a partition $\pi \in P_{o,d}$. A minimal basis partition corresponding to a given successive rank vector $\mathbf{r}$ can be constructed in exactly the same way as the construction of a basis partition for ordinary (unrestricted) partitions, but there are differences in the sequence of steps in the construction which is described in [5]. This construction shows that minimal basis partitions are characterized by the property that no row (in the 2-modular graph) below the Durfee square can be equal (in length) to a column to the right of the Durfee square, and if the last entry $r_k = 0$, then the right hand bottom entry in the Durfee square has to be 1. Although minimal basis partitions are defined by minimality in terms of size, this characterization is in terms of length.

We now define a *basis partition* in $P_{o,d}$ by the property that no row below the Durfee square is equal (in size) to any column to the right of the Durfee square. Thus there is no redundancy (in this sense) in a basis partition. Notice that 2,2,1 as a row is not equal (in size) to 2,2,2 as a column, but both are of equal length. Thus in a basis partition one could have a row 2, 2, 1 below the Durfee square and a column 2,2,2, to the right of the Durfee square, but this is not permissible in a minimal basis partition. Thus while every minimal basis partition is a basis partition, the converse is not true. In the case of unrestricted partitions and their ordinary Ferrers graphs, there is no distinction between minimal basis partitions and basis partitions.

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*Berkovich and Garvan [11] have investigated the rank of partitions in $P_{o,d}$ using 2-modular diagrams, but with rank defined as the length of the first row minus the length of the first column.

The procedure to construct all basis partitions from minimal basis partitions is discussed in [5]. We now consider the generating function of basis partitions and minimal basis partitions, and as a consequence deduce some nice partity results. To facilitate this discussion, we first note that every primary partition is a minimal basis partition and hence a basis partition because the portion below the Durfee square is empty. Also no secondary partition can be a basis partition, and hence cannot be a minimal basis partition as well. Using the sliding operation, basis partitions can be generated nicely from primary partitions. In the case of even columns, the complete block of even columns of a given length has to be slid down. So the generating function of basis partitions in $P_{o,d}$ is

$$\sum_{k=0}^{\infty} \frac{q^{2k^2}(-q^2;q^2)_k}{(q^2;q^2)_k}.(-2q;q^2)_k + \sum_{k=1}^{\infty} \frac{q^{2k^2-1}(-q^2;q^2)_{k-1}}{(q^2;q^2)_{k-1}}.(-2q;q^2)_{k-1}. \tag{7.4}$$

The reason for the presence of $-2q$ in the numerator is because given an odd row (or column), of a specific size, we have exactly two choices of having it as a row or as a column, but not both because in a basis partition there cannot be a redundancy. In fact we can insert parameter z and b in (7.4) with the power of z being the *signature* of a basis partition in $P_{o,d}$ which we define as the number of different parts (even or odd) below the Durfee square, and the power of b being the number of odd parts. More precisely let $b(n;k,j)$ denote the number of basis partitions $\beta \in P_{o.d}$ of n with signature k and $\nu_o(\beta) = j$. We then have

$$\sum_{n,k,j} b(n;k,j) b^j z^k q^n = \sum_{k=0}^{\infty} \frac{q^{2k^2}(-zq^2;q^2)_k(-b(1+z)q;q^2)_k}{(q^2;q^2)_k}$$

$$+ \sum_{k=1}^{\infty} \frac{bq^{2k^2-1}(-zq^2;q^2)_{k-1}(-b(1+z)q;q^2)_{k-1}}{(q^2;q^2)_{k-1}}. \tag{7.5}$$

Analogous to Hirschhorn's result [15] that the Rogers-Ramanujan partitions enumerated with weights which are powers 2 yield the number of basis partitions, we have a similar weighted partition identity here connecting special partitions with basis partitions, and a refinement of it involving the signature. To facilitate its statement, let us rewrite the gap conditions for special partitions $\overline{\pi} : b_1 + b_2 + ... + b_k$ as

$$b_i - b_{i+1} \geq m(b_{i+1}) := 4 + [b_{i+1}]_2, \quad \text{for} \quad 1 \leq i \leq k-1, \tag{7.6}$$

where $[n]_2 = 0$ if n in even, $= 1$ if n is odd. Thus $m(b_{i+1})$ is the minimal permissible value for the difference $b_i - b_{i+1}$ given in terms of the parity of b_{i+1}. We now have

Theorem 9: *For a special partition* $\overline{\pi} : b_1 + b_2 + ... + b_k$ *let its weight be* $w(\overline{\pi}) = 2^{\ell}$, *where*

$$\ell = \{\text{number of gaps}\, b_i - b_{i+1} > m(b_{i+1})\}$$

$$+\{\text{number of gaps}\, b_i - b_{i+1} > m(b_{i+1}) + 2, \text{when}\, b_i \,\text{is odd}\}, \tag{7.7}$$

with the convention $b_{k+1} = -2$. *If* $b(n)$ *is the number of basis partitions of* n *in* $P_{o,d}$, *then*

$$b(n) = \sum_{\overline{\pi}\in\Psi,\sigma(\pi)=n} w(\overline{\pi}). \tag{7.8}$$

More generally, if $b(n, j)$ *is the number of basis partitions in* $P_{o,d}$ *with signature* j, *then*

$$\sum_j b(n; j) z^j = \sum_{\pi\in\Psi,\sigma(\pi)=n} w_z(\overline{\pi}), \tag{7.9}$$

where $w_z(\pi) = (1 + z)^{\ell}$, *with* ℓ *defined as above.*

Note that when $z = -1$ the series on the left in (7.5) collapses to

$$\sum_{k=0}^{\infty} q^{2k^2} + b \sum_{k=1}^{\infty} q^{2k^2-1}.$$

This yields a nice parity result valid with a parameter b. To state this, let

$$B_e(n, b) = \sum_{\beta\in P_{o,d},\sigma(\beta)=n,\psi(\beta)=\,even} b^{\nu_o(\beta)}, \tag{7.10}$$

and

$$B_o(n, b) = \sum_{\beta\in P_{o,d},\sigma(\beta)=n,\psi(\beta)=\,odd} b^{\nu_o(\beta)}. \tag{7.11}$$

In (7.10) and (7.11) the sums are over basis partitions β of n, and $\psi(\beta)$ is the signature of β. Then the interpretation of the collapse of (7.5) when $z = -1$ is given by

Theorem 10:

$$B_e(n,b) - B_o(n,b) = 1 \text{ if } n = 2k^2, \quad b \text{ if } n = 2k^2-1, \quad \text{and } 0 \text{ otherwise.}$$

Compared to Theorem 8, in Theorem 10 we get lacunarity even with a parameter b.

The generating function of minimal basis partitions can also be determined. To facilitate this discussion, we will refer to the situtation where the Durfee square has a 1 in it as Case 1, and the situation where the Durfee square has all twos as Case 2. Note that in Case 1 with a $k\times k$ Durfee square, for a certain j between 1 and $k-1$, if a collection of parts all equal to $2j$ are represented as a set of columns to the right of the Durfee square or as a set of rows below the Durfee square, and if the integer $2j-1$ is to included in the graph, then it has to be placed alongside the collection of $2j$. There is no choice as to where to place the $2j-1$ if the $2j$ occur. But if $2j$ does not occur, we could have $2j-1$ represented either as a row below the Durfee square or as a column to the right of the square, and thus have two choices. With regard to Case 2, the above observations all hold for $1 \le j \le k-1$. In addition, we need to note that the last entry in the successive rank vector has to be non-zero, thereby forcing the graph to have either a row of *length* k below the Durfee square or a column of *length* k to the right of the square, but not both. This row or column of length k could represent either $2k-1$ or $2k$. With these observations, we can modify (7.4) and write down the generating function of $b_m(n)$, the number of minimal basis partitions of n:

$$\sum_{n=0}^{\infty} b_m(n)q^n = 1 + \sum_{k=1}^{\infty} q^{2k^2-1}\prod_{j=1}^{k-1}\{1+2q^{2j-1}+\frac{2q^{2j}(1+q^{2j-1})}{(1-q^{2j})}\}$$

$$(7.12)\quad +\sum_{k=1}^{\infty} q^{2k^2}\prod_{j=1}^{k-1}\{1+2q^{2j-1}+\frac{2q^{2j}(1+q^{2j-1})}{(1-q^{2j})}\}.(2q^{2k-1}+\frac{2q^{2k}(1+q^{2k-1})}{(1-q^{2k})}).$$

As in the case of the generating function of basis partitions, we can refine (7.12), but here it is best to keep track of the number of different lengths below the Durfee square, which we call as *ℓ-signature*, which we will keep track by a parameter ζ. Thus we have the following refinement of (7.12):

$$\sum_{n,j} b_m(n;j)\zeta^j q^n = 1+\sum_{k=1}^{\infty} q^{2k^2-1}\prod_{j=1}^{k-1}\{1+(1+\zeta)q^{2j-1}+\frac{(1+\zeta)q^{2j}(1+q^{2j-1})}{(1-q^{2j})}\}+$$

(7.13)

$$\sum_{k=1}^{\infty} q^{2k^2}\prod_{j=1}^{k-1}\{1+(1+\zeta)q^{2j-1}+\frac{(1+\zeta)q^{2j}(1+q^{2j-1})}{(1-q^{2j})}\}.[(1+\zeta)q^{2k-1}+\frac{(1+\zeta)q^{2k}(1+q^{2k-1})}{(1-q^{2k})})],$$

where $b_m(n;j)$, the number of minimal basis partitions of n with ℓ-signature equal to j.

Finally we note that (7.13) collapses to

$$1+\sum_{k=1}^{\infty} q^{2k^2-1}$$

when $\zeta=-1$. To interpret this collapse, we denote by $\lambda(\mu)$ the ℓ-signature of a minimal basis partition μ, and by

(7.14)

$$M_e(n)=\sum_{\mu\in P_{o,d},\sigma(\mu)=n,\lambda(\mu)=\,even} 1, \quad \text{and} \quad M_o(n)=\sum_{\mu\in P_{o,d},\sigma(\mu)=n,\lambda(\mu)=\,odd} 1.$$

Then the collapse of (7.13) when $z=-1$ has the following interpretation:

Theorem 11:

$$M_e(n)-M_o(n)=1 \text{ if } n=2k^2-1, \quad \text{and 0 otherwise.}$$

A more detailed discussion of the ideas in this section with proofs can be found in [5].

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