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Partial Theta Identities of Ramanujan, Andrews, and Rogers-Fine Involving the Squares

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Abstract. We discuss three partial theta identities involving the squares and cast them in the form of weighted partition identities. The first partial theta identity is due to Ramanujan, and the second due to Andrews. The third is a special case of the famous Rogers-Fine identity. We provide here a description of our combinatorial and analytic approach to all three identities, as well as a comparison of the combinatorial proofs of Berndt-Kim-Yee of Ramanujan's identity and of Chen-Liu of two of our weighted partition theorems. We establish duals and companions of a partition theorem of Berndt-Kim-Yee.

Keywords. Ramanujan's lost notebook, Andrews' partial theta identity, Ramanujan's partial theta identity, partitions, Rogers-Fine identity, weighted partitions.

Subject Classification: 05A15, 05A17, 05A19

1. Introduction

Euler, the founder of the theory of partitions and q-series, established a number of beautiful results, one of the most fundamental being the Pentagonal Numbers Theorem:

$$\prod_{m=1}^{\infty}(1-q^m) = \sum_{k=-\infty}^{\infty}(-1)^k q^{(3k^2-k)/2}. \tag{1.1}$$

Our goal here is to discuss, both analytically and combinatorially, the following three remarkable partial theta identities:

$$1+\sum_{n=1}^{\infty}\frac{(-a)^n q^{n(n+1)/2}(-q)_{n-1}}{(aq^2;q^2)_n} = \sum_{k=0}^{\infty}(-a)^k q^{k^2}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} q^{2n}(q^{2n+2};q^2)_{\infty}(aq^{2n+1};q^2)_{\infty} = \sum_{k=0}^{\infty}(-a)^k q^{k^2}, \tag{1.3}$$

$$f(-1,b,-b;q) = 1+2\sum_{k=1}^{\infty}(-1)^k b^{2k} q^{k^2}, \tag{1.4}$$

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where

$$f(a,b,c;q) = 1 + \sum_{n=1}^{\infty} \frac{(1-a)(abq)_{n-1}bc^n q^n}{(bq)_n}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{b^n c^n q^{n^2}(1-a)(abq)_{n-1}(acq)_{n-1}(1-abcq^{2n})}{(bq)_n(cq)_n}. \quad (1.5)$$

In (1.1)–(1.5) and in what follows, we adopt the standard notation

$$(a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad (1.6)$$

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{j=0}^{\infty}(1-aq^j), \quad \text{when} \quad |q| < 1. \quad (1.7)$$

When the base is q, then as in (1.6) and (1.7) we simply write $(a)_n$ and $(a)_\infty$, but when the base is anything other than q, it will always be displayed as in (1.2).

Identity (1.2) is in Ramanujan's Lost Notebook ([14], p. 38). Identity (1.3) is due to Andrews who had previously established the case $a = -1$ [5,4], but then he noticed that the general identity (1.3) with the parameter a holds when he saw my partition theoretic interpretation of (1.2) – see Theorem 1 below. Identity (1.5) is my variant [2] of the famous Rogers-Fine identity (see (8.6) in Section 8). I was motivated to investigate (1.5) since it was so naturally linked to the partition function. Clearly (1.4) is a special case of (1.5). One reason the three identities are so interesting is because lacunarity holds even with a parameter.

In earlier papers [1–3], we investigated all three partial theta series in detail. More specifically, in [1], we proved (1.2) q-theoretically and provided a weighted partition interpretation (see Theorem 1 in Section 2). In [3] we proved (1.3) and related identities q-theoretically, interpreted (1.3) as a weighted partition theorem (see Theorem 3 in section 2) and the links with Theorem 1. In [2] we proved (1.5) combinatorially and from this showed that (1.4) is equal to a weighted partition theorem (see Theorem 4 in Section 2) for which we supplied a new combinatorial proof.

The most elegant and famous proof of (1.1) is by Fabian Franklin (see [4]) through his fundamental involution on partitions into distinct parts. Chen and Liu [10] have recently constructed an involution which simultaneously yields a bijective proof of both our Theorems 1 and 3 and hence of (1.2) and (1.3). This construction of Chen and Liu is quite intricate and is described in section 7.

In the course of providing combinatorial proofs of several identities in Ramanujan's Lost Notebook, Berndt, Kim, and Yee [7], supply a bijective proof of (1.2) by interpreting it in terms of certain vector partitions. The earlier proof of (1.2) in ([6], p. 25) is analytic. Among all the combinatorial proofs given in [7], the most difficult is that of (1.2). It is noted in [7] that the combinatorial proof of (1.2) does not yield a bijective proof of our Theorem 1, which is the partition version of (1.2). This is what has been accomplished by Chen and Liu [10].

The purpose of this paper is provide a survey of all this recent work on the three partial theta series and to compare the proofs – both q-theoretic and combinatorial. In doing so, we shall simplify the approach of Berndt, Kim and Yee, and in that process obtain some new companions and duals of a certain result in [7]. With regard to (1.4), we point out that certain partial theta series connected with the Rogers-Fine identity have played a role in recent important work on Ramanujan's mock theta functions [9]. This is why we included a discussion of (1.4) here, besides the reason that the partial theta series (1.4) also deals with the squares and has a natural partition interpretation.

In the next section we state the partition theoretic interpretation of the three partial theta identities. For this purpose, we close this section with some notation for partitions.

Throughout we shall use the following notation for various statistics involving partitions: A partition of an integer is a representation of that integer as a sum of positive integers, two such representations considered the same if they differ only in the order of the parts (= summands). For any partition π, we denote by

$$\sigma(\pi) = \text{the sum of the parts of } \pi,$$

$$\lambda(\pi) = \text{the largest part of } \pi,$$

$$\ell(\pi) = \text{the least part of } \pi,$$

$$\nu(\pi) = \text{the number of parts of } \pi.$$

If we wish to count the number of parts with restrictions, we denote this by a subscript. For example $\nu_e(\pi)$ (resp. $\nu_o(\pi)$) denote the number of even (resp. odd) parts of π. Also, $\nu_d(\pi)$ will denote the number of different parts of π. Finally we denote by $P_{d,o}$ the set of partitions into distinct parts with smallest part odd.

2. Partition versions of the identities

Euler's pentagonal series identity (1.1) is equivalent to the following partition theorem:

Theorem E. *Let $Q_e(n)$ (resp. $Q_o(n)$) denote the number of partitions of n into distinct parts, and having an even (resp. odd) number of parts. Then*

$$Q_e(n) - Q_o(n) = (-1)^k \quad \textit{if} \quad n = \frac{3k^2 - k}{2}, \quad \textit{and} \quad 0, \textit{otherwise}. \tag{2.1}$$

To state the partition theoretic version of (1.2), we define certain weights on partitions.

Consider a partition $\pi = b_1 + b_2 + \cdots + b_\nu$, with b_ν being odd. With the convention that $b_{\nu+1} = 0$, define the weight of the $i - th$ gap between b_i and b_{i+1} to be $\omega_i = \omega_i(\pi)$, where

$$\omega_i = a^{\delta_i}, \quad \delta_i = \text{the least integer} \geq (b_i - b_{i+1})/2. \tag{2.2}$$

The weight $\omega_A(\pi)$ of the partition π is defined multiplicatively as

$$\omega_A(\pi) = (-1)^{\nu(\pi)} \prod_{i=1}^{\nu} \omega_i. \tag{2.3}$$

With these weights, we have Ramanujan's identity to be equivalent to:

Theorem 1.

$$\sum_{\pi \in P_{d,o}, \sigma(\pi)=n} \omega_A(\pi) = (-a)^k, \quad \textit{if} \quad n = k^2, \quad \textit{and} \quad 0, \textit{ otherwise.}$$

Theorem 1 was proved in [1] and we will recall that proof here in Section 3 since the ideas will be used in subsequent sections. It has the following striking special case when $a = 1$:

Theorem 2. *Let $R_e(n)$ (resp. $R_o(n)$) denote the number of partitions of n into distinct parts such that the least part is odd, and the number of parts is even (resp. odd). Then*

$$R_e(n) - R_o(n) = (-1)^k, \quad \textit{if} \quad n = k^2, \quad \textit{and} \quad 0, \textit{ otherwise.}$$

Theorem 2 is on par with Euler's Pentagonal Numbers Theorem. Despite its simplicity and elegance, it somehow escaped attention for so many years. Fine actually came quite close to Theorem 2 because in his famous paper [11] he noted that

$$p_{d,o}(n) \quad \text{is odd precisely when} \quad n = k^2, \tag{2.4}$$

which is a consequence of Theorem 2 because $p_{d,o}(n) = R_e(n) + R_o(n)$. In a comprehensive survey that was written in 2003 but published in 2006, Pak [13] raised the problem (see [13], p. 35) of finding an involution that would establish (2.4). In 2004, Bessenrodt and Pak [8] found such an involution which actually established Theorem 2 in a stronger form that is equivalent to our Theorem 3 below, but their involution does not yield Theorem 1.

At first glance, Andrews' identity (1.3) seems to be dealing with partitions into distinct parts with smallest part even, but it is really dealing with partitions $\pi \in P_{d,o}$, namely those with smallest part odd, as shown in [2]. More precisely, (1.3) is equivalent to:

Theorem 3. *For a partition $\pi \in P_{d,o}$, define its weight by*

$$\omega_a(\pi) = (-1)^{\nu(\pi)} a^{\nu_o(\pi)}. \tag{2.5}$$

Then

$$\sum_{\pi \in P_{d,o}, \sigma(\pi)=n} \omega_a(\pi) = (-a)^k, \quad \textit{if} \quad n = k^2, \quad \textit{and} \quad 0, \textit{ otherwise.}$$

It is remarkable that even though the weights in (2.3) and (2.5) are different, their sums over the partitions of n in $P_{d,o}$, are identical.

The combinatorial interpretation of (1.4) involves unrestricted partitions counted with weights, as given by the following result:

Theorem 4. *For each partition π, let its weight be*

$$\omega(\pi) = (-1)^{\nu(\pi)} 2^{\nu_d(\pi)} b^{\nu(\pi)+\lambda(\pi)}. \tag{2.6}$$

Then

$$\sum_{\sigma(\pi)=n} \omega(\pi) = (-1)^k b^{2k}.2, \quad \text{if} \quad n = k^2, \quad \text{and} \quad 0, \text{ otherwise.}$$

Theorem 4 was proved combinatorially in [2].

We now quickly demonstrate, why Theorems 1, 3 and 4 are combinatorial versions of identities (1.2)–(1.4). But first we look at the special case $a = 1$ in (1.2) and its relationship with Theorem 2.

The generating function of partitions into n distinct parts is

$$\frac{q^{n(n+1)/2}}{(q)_n}. \tag{2.7}$$

In (2.7), we view $n(n+1)/2$ as representing the *minimal partition* into n distinct parts, namely $1 + 2 + \cdots + n$. If we represent this minimal partition by a triangular Ferrers graph, then we view the denominator $(q)_n$ in (2.7) as imbedding columns of length j, for $1 \le j \le n$ into this graph of the minimal partition to generate all partitions into n distinct parts. When $a = 1$, identity (1.2) reduces to:

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_{n-1}(1-q^{2n})} = \sum_{k=0}^{\infty} (-1)^k q^{k^2}. \tag{2.8}$$

The main difference between (2.7) and the n-th summand on the left in (2.8) is that the last factor $(1 - q^n)$ in the denominator of (2.7) has been replaced by $(1 - q^{2n})$ in (2.8). What this means is that in the imbedding of columns of length j into the graph of the minimal partition, the columns of length n alone are imbedded in pairs. This means that

$$\frac{(-1)^n q^{n(n+1)/2}}{(q)_{n-1}(1-q^{2n})} \tag{2.9}$$

is the generating function of partitions into n distinct parts with smallest part odd. Once this observation is made, Theorem 2 is clearly seen to be the combinatorial version of (2.8).

For the case of general a in (1.2), we argue a bit differently. First we view

$$\frac{(-a)^n q^{n(n+1)/2}}{(aq^2; q^2)_n} \tag{2.10}$$

as the generating function of partitions obtained by imbedding every column of length j, for $1 \le j \le n$ in pairs, thereby creating partitions with smallest part odd, and the gap between consecutive parts being odd. These partitions π' are counted with weight $\omega_A(\pi')$, with the weight as in (2.3). Finally, the factor $(-q)_{n-1}$ can be viewed as imbedding at most one column of length j for each $1 \leq j \leq n-1$. These imbeddings, whenever they take place, convert the gap between the parts of π' from odd to even, but the smallest part remains (odd) and unchanged with this imbedding. Thus all partitions π into n distinct parts with smallest part odd are generated this way. Not that this imbedding does not change the weight. This shows that Theorem 1 is the combinatorial version of (1.2).

Andrews' identity (1.3) appears to be the generating function for partitions into distinct parts with smallest part even. But then since the series begins with $n = 0$, Andrews does allow 0 as a part, and this is how he stated in [5] the partition version of (1.2) when $a = -1$:

Theorem 5. *Let $\varepsilon_e(n)$ (resp. $\varepsilon_o(n)$) denote the number of partitions of n into distinct non-negative parts with smallest part even, and having an even (resp. odd) number of even parts. Then*

$$\varepsilon_e(n) - \varepsilon_o(n) = 1, \quad \textit{if} \quad n = k^2, \quad \textit{and} \quad 0, \ \textit{otherwise}.$$

Theorem 5 is not the same as our Theorem 3, but equivalent to it as we now demonstrate: If we have a partition π' of n into distinct parts with smallest part even, then either the smallest part is 0, or it is positive. If the smallest part is even and positive, we can always add 0 to it to get another partition π'' of n of the Andrews type but with opposite parity for the number of even parts. Thus the contribution of these partitions to $\varepsilon_e(n) - \varepsilon_o(n)$ would be 0. Now every partition of n with smallest part 0 and second smallest part even is counted as π''. So we need only consider partitions of n of the Andrews type with 0 as the smallest part and second smallest part odd. If we remove 0 from such a partition, we get a regular partition of n into distinct parts with smallest part odd. Thus the only partitions that contribute to the difference $\varepsilon_e(n) - \varepsilon_o(n)$ in Theorem 4 are the partitions of n into distinct parts with smallest part odd. Now in any partition, the parity of the number of odd parts is the parity of n. Thus for any partition π,

$$(-1)^{\nu(\pi)} = (-1)^{\sigma(\pi)}(-1)^{\nu_e(\pi)}.$$

This explains the equivalence of Theorems 3 and 5 and the presence of $(-1)^k$ in Theorem 2.

We now show q-theoretically why Theorem 3 is equivalent to (1.3).

Suppose we define $\omega_a(\pi)$ as in (2.5) for partitions π in the set P_d of ALL partitions into distinct parts. Then it follows that

$$\sum_{\pi \in P_d; \sigma(\pi) \geq 0} \omega_a(\pi) = (q^2; q^2)_\infty (aq; q^2)_\infty. \tag{2.11}$$

Note that the summand corresponding to $n = 0$ on the left in (1.3) is exactly the product on the right in (2.11), and so is the generating function of ALL partitions π into distinct parts counted with weight $\omega_a(\pi)$. Now denote by $P_{d.e}$ the set of all partitions into distinct parts with smallest part *even*. If these partitions are counted with weight $\omega_a(\pi)$, then their generating function is

$$\sum_{\pi\in P_{d,e};\sigma(\pi)\geq 1}\omega_a(\pi)=\sum_{n=0}^{\infty}-q^{2n}(q^{2n+2};q^2)_\infty(aq^{2n+1};q^2)_\infty. \tag{2.12}$$

Observe that the sum in (1.3) for $n\geq 1$ is the negative of the expression on the right in (2.12). This means that on the left in (1.3) the generating function of the partitions in $P_{d,e}$ with the same weights are subtracted from the weighted generating function of ALL partitions in P_d. Thus (1.3) is really the identity

$$\sum_{\pi\in P_{d,o};\sigma(\pi)\geq 0}\omega_a(\pi)q^{\sigma(\pi)}=\sum_{k=0}^{\infty}(-a)^kq^{k^2}. \tag{2.13}$$

The interpretation of (2.13) is clearly Theorem 3.

Remarks. In the same paper [11] where Fine noted (2.4), he established two beautiful companions to Theorem E. To motivate Fine's results, we note that another of Euler's simple but fundamental result on partitions is:

Euler's Theorem. *The number of partitions of an integer n into odd parts is equal to the number of partitions of n into distinct parts.*

There is also the fundamental duality connecting the largest part of a partition and the number of parts by conjugation of Ferrers graphs. If we keep Euler's theorem and this duality in mind, we can better appreciate the following two results of Fine:

Theorem F. *Let $Q_a(n)$ denote the number of partitions of n into distinct parts such that the largest part is $\equiv a(\text{mod } 2)$, $a=0,1$.*

Let $Q_b^*(n)$ denote the number of partitions of n into odd parts such the largest part is $\equiv b(\text{mod } 4)$, $b=1,3$.

Then

$$Q_0(n)-Q_1(n)=(-1)^n\{Q_1^*(n)-Q_3^*(n)\} \tag{2.14}$$

and

$$\begin{aligned}
&Q_0(n)-Q_1(n)=1, \quad \textit{if} \quad n=\frac{3k^2+k}{2}, \quad k\geq 0,\\
&Q_0(n)-Q_1(n)=-1, \quad \textit{if} \quad n=\frac{3k^2-k}{2}, \quad k>0, \quad \textit{and}\\
&Q_0(n)-Q_1(n)=0, \quad \textit{otherwise}.
\end{aligned} \tag{2.15}$$

In [11] Fine stated the analytic versions of these results, namely

$$1+\sum_{n=1}^{\infty}(-1)^n(-q)_{n-1}q^n=1+\sum_{n=1}^{\infty}\frac{(-1)^nq^{2n-1}}{(-q;q^2)_n}=\sum_{k=0}^{\infty}q^{(3k^2+k)/2}-\sum_{k=1}^{\infty}q^{(3k^2-k)/2}. \tag{2.16}$$

The series on the right in (2.16) is neither a theta series nor a partial theta series; it is a *false theta series*.

In Ramanujan's Lost Notebook the following two identities may be found ([14], p. 31):

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q)_{2n}} = \sum_{k=0}^{\infty} q^{12k^2+k}(1-q^{22k+11}) + q\sum_{k=0}^{\infty} q^{12k^2+7k}(1-q^{10k+5}), \qquad (2.17)$$

and

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q)_{2n+1}} = \sum_{k=0}^{\infty} q^{12k^2+5k}(1-q^{14k+7}) + q^2\sum_{k=0}^{\infty} q^{12k^2+11k}(1-q^{2k+1}). \qquad (2.18)$$

If we replace q by q^2 in (2.17) and (2.18), then the left hand sides are actually

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(-q^2;q^2)_{2n}} = \sum_{n=0}^{\infty} \{Q_1^*(2n) - Q_3^*(2n)\} q^{2n},$$

and

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(-q^2;q^2)_{2n+1}} = \sum_{n=0}^{\infty} \{Q_1^*(2n+1) - Q_3^*(2n+1)\} q^{2n}.$$

Thus Fine's results are implicit in Ramanujan's Lost Notebook.

It is to be noted that Fine's results deal with the parity split of the largest part. In contrast, the emphasis in (1.2), (1.3), and in Theorems 1, 2, 3, and 5, is on the parity of the smallest part.

In [1] and [3], we gave q-theoretic proofs of (1.2) and (1.3) and they will be recalled in the next section, since this will be relevant in later sections. To understand why Theorem 4 is equivalent to (1.4), we need to combinatorially interpret (1.5), and so this is postponed to the last section.

3. q-hypergeometric proofs

We begin with the proof of (1.2).

One version of the q-binomial theorem is

$$\frac{1}{(aq;q)_n} = \sum_{j=0}^{\infty} a^j q^j \begin{bmatrix} n+j-1 \\ j \end{bmatrix}_q, \qquad (3.1)$$

where the q-binomial coefficients are defined by

$$\begin{bmatrix} i \\ j \end{bmatrix}_q = \frac{(q)_i}{(q)_j (q)_{i-j}}. \qquad (3.2)$$

If we replace $q \mapsto q^2$ in (3.1) and substitute the resulting expansion of $1/(aq^2;q^2)_n$ into the left hand of (1.2), then we get

$$\sum_{n=1}^{\infty} \frac{(-a)^n q^{n(n+1)/2}(-q)_{n-1}}{(aq^2;q^2)_n} = \sum_{n=1}^{\infty} (-a)^n q^{T_n}(-q)_{n-1} \sum_{j=0}^{\infty} a^j q^{2j} \begin{bmatrix} n+j-1 \\ j \end{bmatrix}_{q^2}, \qquad (3.3)$$

where $T_k = k(k+1)/2$ is the k-th triangular number. In view of (3.2), we may rewrite (3.3) as

$$\sum_{n=1}^{\infty} \frac{(-a)^n q^{n(n+1)/2}(-q)_{n-1}}{(aq^2;q^2)_n} = \sum_{n=1}^{\infty}(-a)^n q^{T_n} \sum_{j=0}^{\infty} \frac{a^j q^{2j}}{(q^2;q^2)_j} \frac{(q^2;q^2)_{n+j-1}}{(q)_{n-1}}. \quad (3.4)$$

Now set $m = n + j$ to convert (3.4) to

$$\sum_{n=1}^{\infty} \frac{(-a)^n q^{n(n+1)/2}(-q)_{n-1}}{(aq^2;q^2)_n} = \sum_{m=1}^{\infty}(-a)^m (q^2;q^2)_{m-1} \sum_{j=0}^{m-1} \frac{(-1)^j q^{2j}}{(q^2;q^2)_j} \cdot \frac{q^{T_{m-j}}}{(q)_{m-j-1}}. \quad (3.5)$$

Thus from (3.5) we see that proving (1.2) is equivalent to establishing

$$\sum_{j=0}^{m-1} \frac{(-1)^j q^{2j}}{(q^2;q^2)_j} \cdot \frac{q^{T_{m-j}}}{(q)_{m-j-1}} = \frac{q^{m^2}}{(q^2;q^2)_{m-1}}. \quad (3.6)$$

At this stage we note that

$$\frac{q^{m^2}}{(q^2;q^2)_{m-1}} \quad (3.7)$$

is the generating function of partitions into m distinct odd parts with smallest part 1. This is realized by arguing exactly as we did in (2.7) and (2.8): We view $m^2 = 1 + 3 + \cdots + (2m-1)$ as the minimal partition into m distinct odd parts. We view $1/(q^2;q^2)_{m-1}$ as a Ferrers graph whose columns have 2 at every node and the lengths of the columns being $\leq (m-1)$. If we imbed these columns into the minimal partition, we get all partitions into m distinct odd parts, but the smallest part remains as 1.

With this partition interpretation of the expression in (3.7) we see that

$$\sum_{m=1}^{\infty} \frac{z^m q^{m^2}}{(q^2;q^2)_{m-1}} = zq(-zq^3;q^2)_\infty. \quad (3.8)$$

At this stage we use a trick, namely the identity

$$zq(-zq^3;q^2)_\infty = \frac{zq(-zq^3;q^2)_\infty(-zq^2;q^2)_\infty}{(-zq^2;q^2)_\infty} = \frac{zq(-zq^2)_\infty}{(-zq^2;q^2)_\infty}. \quad (3.9)$$

The term $zq(-zq^2)_\infty$ is the generating function of partitions into distinct parts with smallest part 1. The well known expansion of $(-zq)_\infty$, the generating function of partitions into distinct parts is

$$(-zq)_\infty = \sum_{k=0}^{\infty} \frac{z^k q^{k(k+1)/2}}{(q)_n} \quad (3.10)$$

Thus arguing as we did to get (3.8), we now have

$$zq(-zq^2)_\infty = \sum_{l=1}^{\infty} \frac{z^l q^{T_l}}{(q)_{l-1}} \quad (3.11)$$

The term $1/(-zq^2;q^2)_\infty$ is the generating function of partitions π into even parts counted with weight $(-z)^{\nu(\pi)}$. Using the well known expansion

$$\frac{1}{(zq)_\infty} = \sum_{n=0}^{\infty} \frac{z^n q^n}{(q)_n} \tag{3.12}$$

and the replacement $q \mapsto q^2$, $z \mapsto -z$ in (3.12), we have

$$\frac{1}{(-zq^2;q^2)_\infty} = \sum_{j=0}^{\infty} \frac{(-1)^j z^j q^{2j}}{(q^2;q^2)_j}. \tag{3.13}$$

Thus (3.11), (3.13), (3.8) and (3.9) yield

$$\sum_{m=1}^{\infty} \frac{z^m q^{m^2}}{(q^2;q^2)_{m-1}} = zq(-zq^3;q^2)_\infty = \left\{\sum_{l=1}^{\infty} \frac{z^l q^{T_l}}{(q)_{l-1}}\right\} \left\{\sum_{j=0}^{\infty} \frac{(-1)^j z^j q^{2j}}{(q^2;q^2)_j}\right\}. \tag{3.14}$$

Finally by comparing the coefficients of z^m on the left hand side and the right hand side of (3.14), we get (3.6) and this proves (1.2).

Next we recall the proof of (1.3) in [3]. For this we will utilize (3.10) to expand the $(aq^{2n+1};q^2)_\infty$ term in (1.3) by replacing $q \mapsto q^2$ and taking $z = aq^{2n}$ in (3.10). This gives

$$\begin{aligned}\sum_{n=0}^{\infty} q^{2n}(q^{2n+2};q^2)_\infty (aq^{2n+1};q^2)_\infty &= \sum_{n=0}^{\infty} q^{2n}(q^{2n+2};q^2)_\infty \sum_{k=0}^{\infty} \frac{(-1)^k a^k q^{k^2+2kn}}{(q^2;q^2)_k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k a^k q^{k^2}}{(q^2;q^2)_k} \sum_{n=0}^{\infty} q^{2n(k+1)}(q^{2n+2};q^2)_\infty.\end{aligned} \tag{3.15}$$

If we compare the coefficients of a^k on the right hand sides of (1.3) and (3.15), we see that to prove (1.3), we need to show that

$$\sum_{n=0}^{\infty} q^{2n(k+1)}(q^{2n+2};q^2)_\infty = (q^2;q^2)_k. \tag{3.16}$$

Replace q^2 by q in (3.16) and rewrite it in equivalent form as

$$\sum_{n=0}^{\infty} q^{n(k+1)}(q^{n+1})_\infty = (q)_k. \tag{3.17}$$

To realize (3.17), divide both sides by $(q)_\infty$ to rewrite it as

$$\sum_{n=0}^{\infty} \frac{q^{n(k+1)}}{(q)_n} = \frac{1}{(q^{k+1})_\infty}. \tag{3.18}$$

Note that by setting $z = q^k$ in (3.12) we get (3.18) and this proves (1.3).

Remark. The proof given above is that of (1.3) which we have shown to have Theorem 3 as its partition interpretation because the partitions in $P_{d,e}$ are counted with weight 0, which is the redundant part of (1.3). If we wish to write down a q-hypergeometric identity that represents Theorem 3 without any redundancy, then it is

$$\sum_{n=1}^{\infty} -aq^{2n-1}(q^{2n};q^2)_\infty(aq^{2n+1};q^2)_\infty = \sum_{k=1}^{\infty}(-a)^k q^{k^2}. \tag{3.19}$$

In [3] we gave a direct proof of (3.19) and discussed some of its consequences.

4. A companion to Ramanujan's identity

Our proof of (1.2) made use of the trick in (3.9) to rewrite $(-zq^3;q^2)_\infty$ suitably. We now ask what happens if we employ the same trick on $(-zq;q^2)_\infty$, use an expansion like (3.10) as the starting point, and work backwards from it? More precisely, start with

$$(-zq;q^2)_\infty = \sum_{m=0}^{\infty} \frac{z^m q^{m^2}}{(q^2;q^2)_m} \tag{4.1}$$

which we have already seen in (3.7). Analogous to (3.9), by employing the same trick, and using expansions (3.10) and (3.13), we get

$$(-zq;q^2)_\infty = \frac{(-zq)_\infty}{(-zq^2;q^2)_\infty} = \left\{\sum_{l=0}^{\infty} \frac{z^l q^{T_l}}{(q)_l}\right\}\left\{\sum_{j=0}^{\infty} \frac{(-1)^j z^j q^{2j}}{(q^2;q^2)_j}\right\}. \tag{4.2}$$

From here we work backwards.

By comparing the coefficients of z^m on the right hand sides of (4.1) and (4.2), we get

$$\sum_{j=0}^{m} \frac{(-1)^j q^{2j}}{(q^2;q^2)_j} . \frac{q^{T_{m-j}}}{(q)_{m-j}} = \frac{q^{m^2}}{(q^2;q^2)_m}$$

which is equivalent to

$$\sum_{j=0}^{m}(-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} q^{2j+T_{m-j}}(-q)_{m-j} = q^{m^2} \tag{4.3}$$

upon multiplying both sides by $(q^2;q^2)_m$ and using the definition of the q-binomial coefficient in (3.2). Now use (4.3) to create a partial theta identity by multiplying by $(-a)^m$ and summing. That is, (4.3) yields

$$\sum_{m=0}^{\infty}(-a)^m q^{m^2} = \sum_{m=0}^{\infty}(-a)^m \sum_{j=0}^{m}(-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} q^{2j+T_{m-j}}(-q)_{m-j}. \tag{4.4}$$

As before, we set $m = n + j$ in (4.4) to convert it to

$$\sum_{m=0}^{\infty}(-a)^m q^{m^2} = \sum_{n=0}^{\infty}(-a)^n q^{T_n}(-q)_n \sum_{j=0}^{\infty} a^j q^{2j} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2}. \tag{4.5}$$

Finally, use the q-binomial expansion (3.1) to evaluate the inner sum on the right of (4.5) to be

$$\sum_{j=0}^{\infty} a^j q^{2j} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2} = \frac{1}{(aq^2; q^2)_{n+1}}.$$

Then (4.5) yields

$$\sum_{m=0}^{\infty} (-a)^m q^{m^2} = \sum_{n=0}^{\infty} \frac{(-a)^n q^{T_n} (-q)_n}{(aq^2; q^2)_{n+1}}, \tag{4.6}$$

which is a companion to Ramanujan's identity (1.2).

Even though we have obtained the companion (4.6) by working backwards, still we would like to have a direct proof of the equality

$$1 + \sum_{n=1}^{\infty} \frac{(-a)^n q^{T_n} (-q)_{n-1}}{(aq^2; q^2)_n} = \sum_{n=0}^{\infty} \frac{(-a)^n q^{T_n} (-q)_n}{(aq^2; q^2)_{n+1}}, \tag{4.7}$$

In [1] we gave two proofs of the equality (4.7), one by evaluating the difference between the n-th partial sums of the two series and showing that this tends to 0 as $n \to \infty$, and another by sketching a combinatorial argument. Here we shall give a different proof of (4.7) by a mixture of combinatorial and q-theoretic ideas, in the spirit of our arguments (2.11)–(2.13) that showed Theorem 3 to be interpretation of (1.3).

First we observe that as per the discussion related to (2.9) and the interpretation of Theorem 2, we have

$$\frac{(-a)^n q^{n(n+1)/2} (-q)_{n-1} q^n}{(aq^2; q^2)_n} \tag{4.8}$$

to be the generating function of partitions into n distinct parts with smallest part *even*, because the q^n factor says that we have imbedded a column of length n to make the smallest part even. Next, note that the expansion of the term corresponding to $n = 0$ on the right in (4.7) begins with 1 which represents the null partition. So we decompose this term as

$$\frac{1}{1 - aq^2} = 1 + \frac{aq^2}{1 - aq^2}. \tag{4.9}$$

For $n \geq 1$, the n-th term in the series on the right in (4.7) differs from the n-th term on the left by the factor $(1 + q^n)/(1 - aq^{2n+2})$. We utilize the decomposition

$$\frac{1 + q^n}{1 - aq^{2n+2}} = 1 + \frac{aq^{2n+2}(1 + q^n)}{(1 - aq^{2n+2})} + q^n \tag{4.10}$$

along with (4.9) to rewrite the companion series as

$$\sum_{n=0}^{\infty}\frac{(-a)^n q^{T_n}(-q)_n}{(aq^2;q^2)_{n+1}} = 1+\sum_{n=1}^{\infty}\frac{(-a)^n q^{T_n}(-q)_{n-1}}{(aq^2;q^2)_n}+\sum_{m=0}^{\infty}\frac{(-a)^m q^{T_m} aq^{2m+2}(-q)_m}{(aq^2;q^2)_{m+1}}$$
$$+\sum_{n=1}^{\infty}\frac{(-a)^n q^{n(n+1)/2}(-q)_{n-1}q^n}{(aq^2;q^2)_n}$$
$$= 1+\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3, \quad \text{respectively.} \tag{4.11}$$

Observe that we have deliberately changed the index of summation in $\sum_2$ from n to m. Also we can attach the null partition term 1 in (4.9) to $\sum_1$ and start $\sum_2$ at $m = 0$ because that starting term in $\sum_2$ is the $1/(1-aq^2)$ term on the right in (4.9).

Clearly $1 + \Sigma_1$ is Ramanujan's series on the left in (4.7) which is the generating function of partitions in $P_{d,o}$ enumerated with weight $\omega_A(\pi)$ as in Theorem 1. Also from (4.8) we see that Σ_3 is the generating function of partitions in $P_{d,e}$ counted with weight $\omega_A(\pi)$ as defined in (2.3). Note however from the m-th term in Σ_2, a column of twos of length $m+1$ has been imbedded, but in the numerator of the m-th term in Σ_2 we only have $(-1)^m$ and not $(-1)^{m+1}$. Thus Σ_2 also enumerates partitions in $P_{d,e}$ but with weight $-\omega_A(\pi)$, because the parity of the number of parts has changed. Now given a partition in $P_{d,e}$, it is counted in each of Σ_2 and Σ_3 exactly once, but with weights of opposite sign. Thus

$$\sum\nolimits_2+\sum\nolimits_3 = 0, \tag{4.12}$$

and so (4.7) follows from (4.11) and (4.12).

Remarks. Although the companion has the elegance of being a sum for $n \geq 0$, there is a redundancy in it, namely (4.12). Ramanujan's series deals only with partitions in $P_{d,o}$ and so there is no redundancy there. Perhaps Ramanujan preferred the form without any redundancy! The companion identity (4.7) will have a role when we establish duals of a partition result of Berndt-Kim-Yee in Section 6.

5. Combinatorial proof of Ramanujan's identity

In a recent paper [7], Berndt, Kim and Yee have provided a combinatorial proof of (1.2) by interpreting the series in terms of certain vector partitions. In [7] they give combinatorial proofs of a number of identities in Ramanujan's Lost Notebook, and the proof of (1.2) is by far the most difficult among all of theirs. We shall sketch their proof of (1.2) by emphasizing the main idea which will be Lemma 1 below, and show that a considerable simplification in the proof can be gained if we cast Lemma 1 as an analytic identity and prove it by a combination of analytic and combinatorial arguments. Our approach leads to companions and duals of Lemma 1 which in turn yield the companion identity (4.6).

Consider the product

$$q^{n(n+1)/2}(-q)_{n-1} \tag{5.1}$$

in the numerator on the summand corresponding to n in (1.2). If we think of $n(n+1)/2$ as representing the triangular Ferrers graph $1+2+\cdots+n$ and imbed the parts from $(-q)_{n-1}$ as columns of length j for $1 \le j \le n-1$ into the triangular graph, we get the following *standard interpretation*:

$q^{n(n+1)/2}(-q)_{n-1}$ *is the generating function of partitions* π_s *into n parts all distinct such that*

(i) *smallest part is 1,*
(ii) *the difference between the parts is* ≤ 2.

The main idea in [7] is to give a very different interpretation of the term in (5.1) as follows:

Lemma 1 (B-K-Y). *The term* $q^{n(n+1)/2}(-q)_{n-1}$ *is also the generating function of partitions* π *into n distinct parts such that*

(i) *smallest part is 1,*
(ii) $\lambda(\pi) < 2n$, *and*
(iii) *if* $2k-1$ *is the largest odd part, then ALL odd parts* $< 2k-1$ *appear as parts. Equivalently, the gap between consecutive odd parts is exactly* 2.

Remark. The combinatorial proof of this lemma which involves the conversion of π_s to π is very difficult. Once this lemma is established, the combinatorial proof of (1.2) is easily completed as we show now using the procedure given in [7].

Completion of combinatorial proof of (1.2) using Lemma 1. By Lemma 1, (1.2) is the generating function of bi-partitions (π, σ), where

π: *is a partition as in Lemma 1, with (ii) of Lemma 1 interpreted as*

$$\lambda(\pi) < 2\nu(\pi) \tag{5.2}$$

and

σ: *is a partition into even parts such that*

$$\lambda(\sigma) \le 2\nu(\pi). \tag{5.3}$$

The partition π is counted with weight $(-a)^{\nu(\pi)}$ and the partition σ is counted with weight $a^{\nu(\sigma)}$. Next let π_e (resp. σ_e) denote the largest even part of π (resp. σ) with the convention that $\pi_e = 0$ if π is null, and $\sigma_e = 0$ if σ is null. There are now three cases:

Case 1. $\pi_e = 0$ and $\sigma_e = 0$.

In this case $\sigma = \phi$ and π is the partition $1+3+\cdots+(2k-1)$. This yields the term $(-a)^k q^{k^2}$ on the right hand side of (1.2).

Case 2. $\pi_e \ge \sigma_e$.

In this case remove π_e from π and add it to σ to get a new bi-partition (π', σ'). Note that because $\pi_e < 2\nu(\pi)$, we must now have $\sigma'_e \le 2\pi'_e$. Also 1 is the smallest part of π'. Thus (π', σ') satisfies the conditions (5.2) and (5.3) above, but we have $\pi'_e < \sigma'_e$.

Case 3. $\pi_e < \sigma_e$.

In this case remove σ_e from σ and add it to π to create a new bipartition (π', σ'), which satisfies the conditions (5.2) and (5.3) but now we have $\pi'_e \geq \sigma'_e$.

Thus an involution has been created, namely

$$(\pi, \sigma) \mapsto (\pi', \sigma') \tag{5.4}$$

which moves a bipartition in Case 2 to Case 3 and vice versa, but in this process the parity of the number of parts changes and the weight attached to (π, σ) cancels the weight attached to (π', σ'). Thus the only contribution is from Case 1 which yields the partial theta series on the right in (1.2). This proves Ramanujan's identity.

6. Simple proof of Lemma 1, duals, companions

We shall now give a simple proof of Lemma 1 by a combination of analytic and combinatorial arguments. The ideas underlying this simplified proof lead to a dual of Lemma 1 and companion results as well.

Simple proof of Lemma 1. Consider a partition π with n distinct parts, satisfying conditions (i), (ii), and (iii) of Lemma 1. Let π have k odd parts, namely $1, 3, 5, \ldots, (2k-1)$. Represent this by the term q^{k^2}. Note that π has $n-k$ even parts, all distinct, and all $\leq 2n-2$. Represent these by $e_1 > e_2 > \cdots > e_{n-k}$. Remove 2 from e_{n-k}, 4 from $e_{n-k-1}, \ldots,$ and $2(n-k)$ from e_1 to get a partition π'' into $\leq n-k$ even parts each $\leq 2(k-1)$ because $2(n-k)$ was subtracted from e_1. The analytic version of Lemma 1 is

$$q^{n(n+1)/2}(-q)_{n-1} = \sum_{k=1}^{n} q^{k^2} \cdot q^{2T_{n-k}} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_{q^2}. \tag{6.1}$$

In (6.1), the q-binomial term is the generating function of π'', and the $q^{2T_{n-k}}$ term accounts for the first $n-k$ consecutive even positive integers subtracted from the $n-k$ even parts of π. The sum over k in (6.1) is to account for all possible partitions into n distinct parts fitting the conditions of Lemma 1. Thus Lemma 1 is equivalent to (6.1).

To prove (6.1), we divide both sides by $(q^2; q^2)_{n-1}$ and rewrite it as

$$\frac{q^{n(n+1)/2}}{(q)_{n-1}} = \sum_{k=1}^{n} \frac{q^{k^2}}{(q^2; q^2)_{k-1}} \cdot \frac{q^{2T_{n-k}}}{(q^2; q^2)_{n-k}}. \tag{6.2}$$

Note that the left hand side of (6.2) is the generating function of partitions into n distinct parts with smallest part 1. Suppose this partition has k odd parts and $n-k$ even parts. Then the generating function of the sub-partition having k distinct odd parts is $q^{k^2}/(q^2; q^2)_k$, and the generating function of the sub-partition having $n-k$ distinct even parts is $q^{2T_{n-k}}/(q^2; q^2)_{n-k}$. This needs to be summed from $k = 1$ to n as on the right in (6.2) to account for all partitions, and not from $k = 0$ because the smallest part is 1. This proves (6.2) and hence Lemma 1.

A Dual Result. In the above proof, there is one step for which the explanation is not combinatorial, namely rewriting (6.1) as (6.2) by dividing by $(q^2;q^2)_{n-1}$ and performing the cancellations. These cancellations hide and bypass the difficult combinatorics underneath.

The above method leads to a dual to Lemma 1. To realize this, note that the q-binomial coefficient enjoys the property

$$\begin{bmatrix} n-1 \\ n-k \end{bmatrix}_{q^2} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q^2}, \tag{6.3}$$

and so (6.1) is equivalent to

$$q^{n(n+1)/2}(-q)_{n-1} = \sum_{k=1}^{n} q^{2T_{n-k}}.q^{k^2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q^2}. \tag{6.4}$$

In (6.4) the $q^{2T_{n-k}}$ term by itself and not attached to the q-binomial coefficient can be interpreted as saying that the even parts of the partition π are $2, 4, \ldots, 2(n-k)$. This type of condition previously imposed on the odd parts is now imposed on the evens. The q-binomial coefficient in (6.4) is the generating function of partitions into $\leq 2(n-k)$ even parts each $\leq k-1$. Thus (6.4) is equivalent to the following dual of Lemma 1:

Lemma 2. *The expression in (5.1) is the generating function of partitions π into n distinct parts such that*

(i) *smallest part is 1,*
(ii) $\lambda(\pi) < 2n$, *and*
(iii) *if $2(n-k)$ is the largest even part, then ALL even parts $< 2(n-k)$ appear as parts. Equivalently, the gap between consecutive even parts is exactly 2.*

Remark. There is a very nice combinatorial way of realizing the equivalence of Lemmas 1 and 2 by interpreting (6.3) in terms of conjugation of Ferrers graphs. More precisely start with a partition π into n distinct parts satisfying the conditions of Lemma 1 and having k odd parts. Then as per the discussion related to (6.1), π can be decomposed into a triple of partitions (π_1, π_2, π_3), where π_1 is simply $1 + 3 + \cdots + (2k-1)$, π_2 is $2 + 4 + \cdots + 2(n-k)$ and π_3 is a partition into even parts such that

$$\nu(\pi_3) \leq 2(n-k) \quad \text{and} \quad \lambda(\pi_3) \leq 2(k-1). \tag{6.5}$$

The inequalities in (6.5) relate to the q-binomial coefficient in (6.1). Now represent π_1, π_2, π_3 as 2-modular Ferrers graphs. Replace π_3 by its conjugate ${\pi^*}_3$ as a 2-modular graph. Note that

$$\nu({\pi^*}_3) \leq 2(k-1) \quad \text{and} \quad \lambda({\pi^*}_3) \leq 2(n-k). \tag{6.7}$$

We interpret this conjugation as the q-binomial coefficient in (6.4) and

$$q^{k^2} \cdot \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q^2}$$

as representing the imbedding of $\pi^*{}_3$ into π_1, thereby producing a general partition into k distinct odd parts with smallest part 1 and largest part $< 2n$. This yields a partition π' of the type satisfying the conditions of Lemma 2 with k odd parts, smallest part 1 and the even parts being $2, 4, 6, \ldots, 2(n-k)$. The procedure described above converting the partitions π of Lemma 1 to π' of Lemma 2 is reversible. Thus Lemma 2 is equivalent to Lemma 1 and is its dual.

Companions. The ideas underlying (6.1) and (6.2) involving the partitions π satisfying the conditions of Lemma 1 and decomposing them into π_1, π_2, π_3 to get the series in (6.1), would work even if we did not have the condition that the smallest part should be 1. So if we simply consider a partition into n distinct parts, then we first replace (6.2) by

$$\frac{q^{n(n+1)/2}}{(q)_n} = \sum_{k=1}^{n} \frac{q^{k^2}}{(q^2;q^2)_k} \cdot \frac{q^{2T_{n-k}}}{(q^2;q^2)_{n-k}}. \tag{6.8}$$

which is proved in exactly the same way as (6.2). Then working backwards, we write (6.8) in the equivalent form

$$q^{n(n+1)/2}(-q)_n = \sum_{k=1}^{n} q^{k^2} . q^{2T_{n-k}} \begin{bmatrix} n \\ n-k \end{bmatrix}_{q^2}. \tag{6.9}$$

Now (6.9) and (6.8) are the companions to (6.1) and (6.2) respectively. The *standard interpretation* of the product on the left in (6.9) is that

$q^{n(n+1)/2}(-q)_n$ *is the generating function of partitions* π_s *into n parts all distinct such that the difference between the parts is* ≤ 2.

By interpreting the series in (6.9) we have the following companion to Lemma 1:

Lemma 3. *The term* $q^{n(n+1)/2}(-q)_n$ *is also the generating function of partitions* π *into n distinct parts such that*

(i) $\lambda(\pi) \leq 2n$, *and*
(ii) *if* $2k-1$ *is the largest odd part, then ALL odd parts* $< 2k-1$ *appear as parts. Equivalently, the gap between consecutive odd parts is exactly* 2.

Next by taking the conjugate of partitions enumerated by the q-binomial coefficient in (6.8) and following the imbedding idea as above, we get a dual of Lemma 3 which is a companion to Lemma 2, namely:

Lemma 4. *The term* $q^{n(n+1)/2}(-q)_n$ *is also the generating function of partitions* π *into n distinct parts such that*

(i) $\lambda(\pi) \leq 2n$, *and*

(ii) *if* $2(n-k)$ *is the largest even part, then ALL even parts* $< 2(n-k)$ *appear as parts. Equivalently, the gap between consecutive even parts is exactly 2.*

Just as Lemma 1 yielded a combinatorial proof of Ramanujan's identity (1.2), Lemma 3 gives a combinatorial proof of our companion identity (4.6) as we demonstrate now:

Decompose the summand corresponding to n on the right in (4.6) as

$$(-a)^n q^{n(n+1)/2}(-q)_n \cdot \frac{1}{(aq^2;q^2)_{n+1}}. \tag{6.10}$$

Using the decomposition in (6.10), interpret the series on the right in (4.6) as the generating function of bi-partitions (π, σ) such that

π: *is a partition as in Lemma 3, with (i) of Lemma 1 interpreted as*

$$\lambda(\pi) \le 2\nu(\pi) \tag{6.11}$$

and

σ: *is a partition into even parts such that*

$$\lambda(\sigma) \le 2(\nu(\pi)+1). \tag{6.12}$$

Let π_e and σ_e denote the largest even parts of π and σ respectively. As before, we have three cases.

Case 1. $\pi_e = 0$ and $\sigma_e = 0$.

In this case $\sigma = \phi$ and π is either null or is the partition $1+3+\cdots+(2k-1)$. This yields the term $(-a)^k q^{k^2}$ on the right hand side of (1.2) including the case $k=0$.

Case 2. $\pi_e \ge \sigma_e$.

In this case remove π_e from π and add it to σ to get a new bi-partition (π', σ'). Note that because $\pi_e \le 2\nu(\pi)$, we must now have $\sigma'_e \le 2(\pi'_e + 2)$. Thus (π', σ') satisfies the conditions (6.11) and (6.12) above, but we have $\pi'_e < \sigma'_e$.

Case 3. $\pi_e < \sigma_e$.

In this case remove σ_e from σ and add it to π to create a new bipartition (π', σ'), which satisfies the conditions (6.11) and (6.12) but now we have $\pi'_e \ge \sigma'_e$.

Thus an involution has been created, namely

$$(\pi, \sigma) \mapsto (\pi', \sigma') \tag{6.13}$$

which moves a bipartition in Case 2 to Case 3 and vice versa, but in this process the parity of the number of parts changes and the weight attached to (π, σ) cancels the weight attached to (π', σ'). Thus the only contribution is from Case 1 which yields the partial theta series on the left in (4.6).

Remarks.

1) Interestingly, even though Lemmas 2 and 4 are the duals of Lemmas 1 and 3, we do not know of a different partial theta identity emerging from them.

2) Berndt-Kim-Yee observe [7] that while Lemma 1 yields a combinatorial proof of Ramanujan's identity (1.2), it does not provide a combinatorial proof of our weighted partition version of (1.2), namely Theorem 1. Recently Chen and Liu [10] have constructed an involution which simultaneously provides a combinatorial proof of our weighted partition Theorems 1 and 3, and we discuss this next.

7. An involution for the two weighted partition theorems

Chen and Liu [10] have recently established an involution on the set of partitions into distinct parts with smallest part odd that simultaneously provides a combinatorial proof of both our weighted partition Theorems 1 and 3. We will describe the Chen-Liu construction in this section.

Before discussing this involution, we note that for partitions $\pi \in P_{d,o}$, the power of a in $\omega_A(\pi)$ in Theorem 1 is always at least the power of a in $\omega_a(\pi)$ in Theorem 3. More precisely, if we define the exponents $e_A(\pi)$ and $e_a(\pi)$ by

$$\omega_A(\pi) = (-1)^{\nu(\pi)} a^{e_A(\pi)} \quad \text{and} \quad \omega_a(\pi) = (-1)^{\nu(\pi)} a^{e_a(\pi)}, \tag{7.1}$$

then

$$e_A(\pi) \geq e_a(\pi), \quad \text{for all} \quad \pi \in P_{d,o}, \tag{7.2}$$

with equality if and only if $\pi : 1 + 3 + \cdots + (2k-1)$ is the standard partition of k^2. The inequality (7.2) follows at once from the definitions of ω_A and ω_a, because (2.2) implies

$$e_A(\pi) = \sum_{i=1}^{\nu(\pi)} \delta_i \geq \sum_{i=1}^{\nu(\pi)} 1 = \nu(\pi) \geq e_a(\pi) = \nu_o(\pi), \tag{7.3}$$

with equality occurring in (7.3) if and only if π has no even parts and the gaps between the consecutive odd parts starting with 1 are all equal to 2. Under the involution of Chen and Liu, each partition $\pi \in P_{d,o}$ of an integer n is assigned its dual or mate $\pi' \in P_{d,o}$ of the integer n such that

$$\omega_A(\pi) + \omega_A(\pi') = 0 = \omega_a(\pi) + \omega_a(\pi'), \quad \text{if} \quad \pi \neq \pi', \tag{7.4}$$

and

$$\pi = \pi' \quad \text{if and only if} \quad \pi : 1 + 3 + 5 + \cdots + (2k-1) \tag{7.5}$$

which is precisely when equality occurs in (7.2).

Chen and Liu interpret the k-th summand in Ramanujan's partial theta series

$$(-a)^k q^{k(k+1)/2} (-q)_{k-1} \times \frac{1}{(aq^2; q^2)_k} \tag{7.6}$$

as the weighted generating function of partitions in $D_k \times E_k$, where D_k is the set of partitions δ into k distinct parts with smallest part 1 and gap between consecutive parts ≤ 2, and E_k is the set of partitions σ into even parts $\leq 2k$, with the partition δ being counted with weight $(-a)^k$ and the partition σ counted with weight $a^{\nu(\sigma)}$.

Thus the term in (7.6) is the weighted generating function of these bi-partitions $(\delta, \sigma) \in D_k \times E_k$. Notice that for $\delta \in D_k$, Chen and Liu use the *standard interpretation* for the product in (5.1) and not the deeper interpretation of Berndt-Kim-Yee provided by Lemma 1.

The bi-partitions (δ, σ) are in one-to-one correspondence with the partitions $\pi \in P_{d,o}$ by the imbedding process that we used to get the weighted partition interpretation of (1.2) in Theorem 1. The involution of Chen and Liu is on the bi-partitions (δ, σ), but it easily translates to an involution on $\pi \in P_{d,o}$ in view of the imbedding.

Represent δ and σ as 2-modular graphs. Let $\delta_1 > \delta_2 > \cdots > \delta_k$ denote the parts of δ, and $\sigma_1 \geq \sigma_2 \geq \ldots$ denote the parts of σ. Chen and Liu define a statistic called *modular leg hook* in δ as follows: If δ_i is an even part other than the largest part, the modular leg hook H_i consists of all the nodes in the i-th row of the 2-modular graph of δ together with the nodes in the first column above the i-th row of δ. Denote the sum of the nodes on the modular hook H_i by $|H_i|$.

Among the modular hooks H_i, we are interested in those such that the removal of H_i from $\delta \in D_k$ yields a 2-modular graph of a partition $\delta' \in D_{k-1}$. We call such a hook as "acceptable" – this is our terminology, not that of Chen and Liu. Among all acceptable modular leg hooks H_i of δ, consider one that has maximum height (namely with i maximal). We denote such a maximal acceptable modular leg hook by $H(\delta)$, if it exists. Clearly

$$|H(\delta)| \leq 2k - 2. \tag{7.7}$$

Following Chen and Liu, we describe the involution by considering several possibilities.

Type A. There are two cases here.

Case 1. Suppose $H(\delta)$ exists and $|H(\delta)| \geq \sigma_1$. Then remove $H(\delta)$ from δ and add it as a part to σ, thereby creating a bi-partition $(\delta', \sigma') \in D_{k-1} \times E_{k-1}$ in view of (7.7).

Case 2. Either $H(\delta)$ exists and $|H(\delta)| < \sigma_1$, or $H(\delta)$ does not exist and $\delta_1 + 2 < \sigma_1$. In this case remove σ_1 from σ and insert it as the maximal acceptable modular hook into δ. This creates a bi-partition $(\delta', \sigma') \in D_{k+1} \times E_k$. It is to be noted that the insertion of σ_1 into δ as the maximal acceptable leg hook is not at all obvious. But it can be achieved as follows: Let i be the largest positive integer such that $\sigma_1 - 1 - 2i > \delta_{i+1}$. Then add 2 to the first i parts of δ_i and insert $\sigma_1 - 2i$ as a new part of δ just above δ_{i+1}. It follows that after the insertion of σ_1 into δ, the resulting bi-partition (δ', σ') fits into Case 1 above, and so we have an involution.

Even though an involution has been set up, Type A clearly does not exhaust all possibilities. The remaining cases are covered under *Type B*. Suppose $H(\delta)$ does not exist and $\sigma_1 \leq \delta + 2$. Here if δ has even parts, we choose the largest among them and denote it by δ_e. There are two cases:

Case 1. Suppose $\delta_e \geq \sigma_1$. In this case remove δ_e from δ and add it to σ as the largest part. Note that $\delta_e \leq 2k - 2$, and so the resulting bi-partition $(\delta', \sigma') \in D_{k-1} \times E_{k-1}$.

Case 2. Suppose $\delta_e < \sigma_1$. In this case remove σ_1 from σ and add it to δ as a part (not as a modular leg hook!). The resulting bi-partition $(\delta', \sigma') \in D_{k+1} \times E_k$.

As in Type A, here in Type B, the correspondence between (δ, σ) and (δ', σ') is an involution where one passes from Case 1 to Case 2 and vice-versa if $(\delta, \sigma) \neq (\delta', \sigma')$.

Notice that Types A and B cover all possibilities when $(\delta, \sigma) \neq (\delta', \sigma')$. This involution on the bi-partitions, translates to an involution on the partitions $\pi \in P_{d,o}$. That is

$$(\delta, \sigma) \mapsto \pi \in P_{d,o}, \quad (\delta', \sigma') \mapsto \pi' \in P_{d,o}, \tag{7.8}$$

$$(\delta, \sigma) \mapsto (\delta', \sigma'), \quad \pi \mapsto \pi'. \tag{7.9}$$

Under this involution, the number of parts of δ and δ' are of opposite parity, and the number of parts of π and π' are of opposite parity whenever $\pi \neq \pi'$. Notice that in going from (δ, σ) to (δ', σ') and vice-versa, only one even modular leg hook (or part) is moved, and this does not affect the power of a attached to either bi-partition. Thus owing to the imbedding and the correspondence in (7.8), we see that under the involution

$$e_A(\pi) = e_A(\pi'), \quad \text{but} \quad (-1)^{\nu(\pi)} = -(-1)^{\nu(\pi')}, \quad \text{if} \quad \pi \neq \pi'. \tag{7.10}$$

Consequently the first equality in (7.4) holds whenever $\pi \neq \pi'$.

The only remaining case is when σ is empty and δ has no even parts, which means δ is just $1 + 3 + \cdots + (2k - 1)$. Thus these partitions are the only fixed points under the involution. Thus the involution proves Theorem 1 combinatorially.

To realize Theorem 2 through the involution, utilize the correspondence given by (7.9) and observe that under the involution for the bi-partitions, the number of odd parts remains unaffected. That is

$$\nu_o(\pi) = \nu_o(\pi') \quad \text{and so} \quad e_a(\pi) = e_a(\pi'). \tag{7.11}$$

Since the number of parts of π and π' are of opposite parity, we see that the second inequality in (7.4) holds whenever $\pi \neq \pi'$. The case $\pi = \pi'$ is precisely when π is the partition $1 + 3 + \cdots + (2k - 1)$ of k^2, and so the involution proves our Theorem 2 as well.

8. A partial theta identity from the Rogers-Fine identity

Euler's Pentagonal Numbers Theorem, as well as the partial theta identities of Ramanujan and Andrews, all deal with partitions into distinct parts. In this concluding section, we will discuss a partial theta identity involving the squares, namely (1.4), that actually deals with unrestricted partitions. This identity falls out as a special case of a variation of the famous Rogers-Fine identity as was noticed and proved in [2].

The generating function of unrestricted partitions π in which we keep track of $\nu_d(\pi)$ and either $\nu(\pi)$ or $\lambda(\pi)$ but *not* both, has a product representation, namely

$$\sum_{\pi} (1 - a)^{\nu_d(\pi)} z^{\nu(\pi)} q^{\sigma(\pi)} = \frac{(azq)_\infty}{(zq)_\infty}, \tag{8.1}$$

and therefore

$$\sum_{\pi}(1-a)^{\nu_d(\pi)}z^{\lambda(\pi)}q^{\sigma(\pi)} = \frac{(azq)_\infty}{(zq)_\infty}, \tag{8.2}$$

as well, because under conjugation of Ferrers graphs of partitions, $\nu(\pi)$ and $\lambda(\pi)$ are interchanged. However the three variable generating function

$$f(a,b,c;q) := \sum_{\pi}(1-a)^{\nu_d(\pi)}b^{\nu(\pi)}c^{\lambda(\pi)}q^{\sigma(\pi)} \tag{8.3}$$

has only a series representation

$$f(a,b,c;q) = 1 + \sum_{n=1}^{\infty}\frac{(1-a)(abq)_{n-1}bc^nq^n}{(bq)_n} \tag{8.4}$$

but not a product representation. We feel that even though $f(a,b,c;q)$ is so fundamental, the lack of a product representation is one reason it has not been studied closely.

The change of $\nu(\pi)$ into $\lambda(\pi)$ and vice-versa induced by conjugation clearly proves the symmetry property

$$f(a,b,c;q) = f(a,c,b;q), \tag{8.5}$$

but this symmetry is not present in the series representation in (8.4). In an attempt to obtain a series representation for $f(a,b,c;q)$ that renders the symmetry in (8.5) explicit, we studied partitions by their Durfee square representations and that led us to the second series in (1.5) which is symmetric in b and c.

The well-known Rogers-Fine identity in the form obtained by Fine [[11], Eq. (14.1)] is

$$F(\alpha,\beta,\tau;q) := \sum_{n=0}^{\infty}\frac{(\alpha q)_n\tau^n}{(\beta q)_n} = \sum_{n=0}^{\infty}\frac{(\alpha q)_n(\alpha\tau q/\beta)_n\beta^n\tau^nq^{n^2}(1-\alpha\tau q^{2n+1})}{(\beta q)_n(\tau)_{n+1}}. \tag{8.6}$$

Our function f and the Rogers-Fine function F are connected by the relation

$$\frac{(1-bq)}{(1-a)bcq}.\{f(a,b,c;q)-1\} = F(ab,bq,cq;q) \tag{8.7}$$

as can be seen from (8.4), (1.5) and (8.6). If we take

$$c=-b \quad \text{and} \quad a=-1, \tag{8.8}$$

then the second series in (1.5) reduces to

$$1+2\sum_{n=1}^{\infty}(-1)^nb^{2n}q^{n^2}, \tag{8.9}$$

which is identity (1.4). In view of the partition interpretation of $f(a,b,c;q)$ given in (8.3), Theorem 4 is the partition version of (1.4). It is obviously of interest to prove

Theorem 4 combinatorially because we wish to understand how the cancellation of the weights on unrestricted partitions given by (2.6) takes place to yield the partial theta series in (1.4). Such a combinatorial proof of Theorem 4 was given in [2] and we describe the main ideas here.

Given any partition π and its Ferrers graph, we denote its Durfee square by $D(\pi)$, the portion to the right of $D(\pi)$ by π_r, and the portion below $D(\pi)$ by π_b. As in [2], we call a partition π to be *primary* if π_b is empty. A starting fundamental observation in [2] is that the set of all partitions of a given integer N can be obtained by considering all primary partitions of N and then sliding the columns to the right of the Durfee squares of such partitions and placing them below the Durfee square. One of the important invariants under the sliding operation is

$$\lambda(\pi) + \nu(\pi)$$

as well as the size of all the successive hooks of the partition, with each node on the diagonal of the Durfee square being the vertex of a hook. In [2] the following stronger version of Theorem 4 was proved combinatorially:

Theorem 5. *Let π be any given primary partition for which π_r is non-empty. Call such a primary partition non-trivial. Consider the set of all partitions generated by a given non-trivial primary partition (including itself) by performing all possible sliding operations. Then the sum of the weights in (2.6) over all such partitions generated by a non-trivial primary partition is* 0.

In [2] Theorem 5 was proved in a sequence of steps which we briefly recall here:

Step 1. If we have a sum in which the first term is 2, consecutive terms have opposite signs, the absolute value of the last term is 2, and the absolute value of every term other than the first term and the last term is 4, then the sum is 0, provided the sum has at least two terms.

Step 2. In a given nontrivial primary partition, consider the columns of a fixed length ℓ in π_r. Each set of columns of fixed length contributes a corner of π_r. And each corner contributes a factor 2 to the factor $2^{\nu_d(\pi)}$ in the weight $\omega(\pi)$ in (2.6). If a column of length ℓ is slid down and placed as a row below the Durfee square, then there are two possibilities:

(i) There are no more columns of length ℓ in π_r. Thus ν_d has decreased by 1 for the portion which is $D(\pi) + \pi_r$, but we have gained a (different) part by the row(s) below the Durfee square. Thus $2^{\nu_d(\pi)}$ is unchanged, as is the expression in (8.8), but $\nu(\pi)$ has increased by 1. This means the weight $\omega(\pi)$ has changed in sign and the net effect is

$$2 + (-2) = 0.$$

(ii) After sliding a column of length ℓ, we still have a column of length ℓ in π_r. Thus the corner in π_r that originally was there, still remains, but a new part of length ℓ has been added to the Ferrers graph due to the sliding operation. This contributes a factor of 4 to the weight $\omega(\pi)$. However, in this case, sliding more columns of length ℓ only increases $\nu(\pi)$ but $\nu_d(\pi)$ increases by 1 only the very first time a column of length ℓ is slid down, but not later. Thus if there are k columns of length ℓ, the contribution of these columns to the weight $\omega(\pi)$ under the sliding operation is

$$2 - 4 + 4 - 4 \ldots . \pm 4 \mp 2 = 0 \tag{8.10}$$

where the sum in (8.9) has $k + 1 \geq 2$ terms.

Thus the contribution to the weight $\omega(\pi)$ in (2.6) by performing all possible sliding operations on a set of columns of a given length ℓ is the factor 0.

Step 3. **Independence –** Sliding a set of columns of a given length is *independent* of sliding a set of columns of a different length. The interpretation of this is that we would be considering products of expressions as in (8.9) when summing the weights $\omega(\pi)$ over all partitions generated by a given non-trivial primary partition.

In summary, the sum of the weights of all partitions which are born out of non-trivial primary partitions would be zero. This proves Theorem 5. Finally if π is a trivial primary partition, then π_r is empty, and so π is the unique partition that it generates. In this case its weight is

$$2(-1)^{\nu(\pi)+\lambda(\pi)}$$

which is the k-th term in the series in (1.4) if π is the $k \times k$ square of nodes. Thus Theorem 4 follows.

Remarks.

(i) We included a discussion of Theorem 4 here because once again it is the squares alone that survive, thereby yielding the partial theta series in (1.4), and to show how such a partial theta series with squares can be associated naturally with unrestricted partitions instead of partitions into distinct parts.

(ii) Fine's proof of (8.6) (see [[11], Eq. 14.1]) involves the transformation properties of $F(\alpha, \beta, \tau; q)$. Our proof of (1.5) in [2] is purely combinatorial and uses the Durfee square representation of the Ferrers graphs of partitions.

(iii) Fine noticed the striking special case of his identity which yields the partial theta series (8.9) (see [11], Eq. 14.31). Our variant of Fine's identity, namely (1.5), has a natural partition interpretation, and so it enabled us to cast the partial theta identity as the weighted partition result, namely Theorem 4. In going from (1.5) to (1.4) there was significant cancellation of various terms. In spite of these cancellations, it was possible to give a combinatorial proof of (1.4) and hence of Theorem 4, again using Durfee squares, but in a very different manner.

(iv) If we choose $b = -1$ in (1.4), then the partial theta series there becomes a theta series which has a product representation, namely

$$1 + 2\sum_{k=0}^{\infty}(-1)^k q^{k^2} = \sum_{k=-\infty}^{\infty}(-1)^k q^{k^2} = \prod_{m=1}^{\infty}\frac{(1-q^m)}{(1+q^m)}. \tag{8.11}$$

By using the expansion

$$\frac{(1-q^m)}{(1+q^m)} = 1 - \frac{2q^m}{(1+q^m)} = (1 + 2\{-q^m + q^{2m} - q^{3m} + q^{4m} - \ldots\}), \tag{8.12}$$

we can interpret the infinite product in (8.11) as the weighted generating function of unrestricted partitions with weight

$$(-1)^{\nu(\pi)}2^{\nu_d(\pi)}$$

which by conjugation is the same as attaching the weight

$$(-1)^{\lambda(\pi)}2^{\nu_d(\pi)} \tag{8.13}$$

to unrestricted partitions. Notice that when $b = -1$, the weight $\omega(\pi)$ in (2.6) reduces to the weight in (8.13).

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